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PERTURBATION OF DISPERSIVE SHALLOW WATER WAVES

by

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A THESIS

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ABSTRACT

This thesis addresses the dynamics of dispersive shallow water wave that is governed by the Rosenau-KdV equation with power law nonlinearity. The singular 1-soliton solution is derived by the ansatz method. Subsequently, the soliton perturbation theory is applied to obtain the adiabatic parameter dynamics of the water waves. Finally, the integration of the perturbed Rosenau-KdV equation is obtained by the ansatz method as well as the semi-inverse variational principle.

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Chapter 1

INTRODUCTION

The theory of shallow water waves is one of the very important areas of research in ocean dynamics. In this discipline, the study of solitary waves receives a lot of attention[1-20]. Therefore, it is important to focus on solitary waves in a detailed manner from a mathematical point of view. There are several models that describe the dynamics of shallow water waves such as the Korteweg-de Vries (KdV) equation, Boussinesq equation, Benjamin-Bona-Mahoney equation, Kawahara equation, Peregrine equation or the regularized long wave equation and many others [1-20]. Besides these, there are the coupled vector KdV equation, coupled vector Boussinesq equation, Gear-Grimshaw model or the Bona-Chen equation that all model two-layered shallow water waves [2, 3]. These kind of scenarios occur, for example, in an oil spill along an ocean shore like the Exxon Valdez spill at Prince William Sound in Alaska.

This thesis is however going to focus on the dispersive shallow water waves. Again, there are several models that describe these waves. A few of them are the Rosenau-Kawahara equation, sixth order Boussinesq equation, Rosenau-KdV (R-KdV) equation. This paper is going to shine some light on the R-KdV equation with power law nonlinearity in the presence of several perturbation terms that arise due to numerous natural phenomena along ocean shores and beaches. The integrability aspect of this equation will be addressed along with the conservation laws and soliton perturbation theory. The integration phenomena will lead to several constraint conditions that will naturally fall out during the course of derivation of the soliton solution.

Chapter 2

GOVERNING EQUATION

The dimensionless form of the R-KdV equation that is going to be studied in this paper is given by [7]

$$q_t + aq_x + bq_{xxx} + cq_{xxxxt} + k\left(q^n\right)_x = 0 \tag{2.1}$$

Here in (2.1), the dependent variable $q(x, t)$ represents the shallow water wave profile while the independent variables x and t represent the spacial and temporal variables, respectively. The coefficient of a represents the drifting term, the coefficient of b is the third order dispersion and the coefficient of c represents the higher order dispersion term. Finally, the last term k is the nonlinear term where n is the power law nonlinearity parameter and it represents the strength of nonlinearity. However, the restriction is that

$$n \neq 0, 1 \tag{2.2}$$

There are actually two types of solitary wave solutions that will be considered in this paper. The first one was already derived earlier in 2011 [7]. Subsequently, the singular solitary wave solution will be derived using the ansatz method. These two are detailed in the next two subsections.

2.1 Solitary Wave Solution

The solitary wave solution or 1-soliton solution of the R-KdV equation is given by [7, 19]

$$q(x,t) = A\,\mathrm{sech}^{\frac{4}{n-1}}\left[B(x-vt)\right] \tag{2.3}$$

where

$$n > 1 \tag{2.4}$$

The relation between the soliton amplitude (A) and the inverse width (B) is given by

$$A = \left[\frac{2(n+1)(n+3)(3n+1)bB^2}{k(n-1)^2\left(n^2+2n+5\right)}\right]^{\frac{1}{n-1}} \tag{2.5}$$

and the velocity (v) of the soliton is

$$v = \frac{a(n-1)^4 + 16bB^2(n-1)^2}{(n-1)^4 + 256cB^4} \tag{2.6}$$

or

$$v = \frac{b(n-1)^2}{4cB^2\left(n^2+2n+5\right)} \tag{2.7}$$

Upon equating the two expressions of the velocity of the soliton leads to the constraint condition

$$b(n-1)^4 = 4cB^2\left\{16(n+1)^2bB^2 + a(n-1)^2\left(n^2+2n+5\right)\right\} \tag{2.8}$$

and

$$bk > 0 \tag{2.9}$$

for odd values of n. Now, from (2.8), the soliton width (B) can be expressed as

$$B = \frac{n-1}{n+1}\left[\frac{-ac\left(n^2+2n+5\right)+\sqrt{a^2c^2\left(n^2+2n+5\right)^2+16b^2c(n+1)^2}}{32bc}\right]^{\frac{1}{2}} \tag{2.10}$$

Substituting the value of B from (2.10) into (2.5) leads to

$$\begin{aligned} A &= \left[-ac\left(n^2+2n+5\right)+\sqrt{a^2c^2\left(n^2+2n+5\right)^2+16b^2c(n+1)^2}\right]^{\frac{1}{n-1}} \times \\ &\times \left[\frac{(n+3)(3n+1)}{16ck(n+1)\left(n^2+2n+5\right)}\right]^{\frac{1}{n-1}} \end{aligned} \tag{2.11}$$

which is the amplitude of the soliton. It is easily observed from (2.10) and (2.11) that the soliton amplitude and the width are guaranteed to exist provided

$$c > 0 \tag{2.12}$$

It needs to be noted that these exact values of the soliton amplitude and width, as given by (2.11) and (2.10), respectively, were not reported by Esfahani in 2011 and is thus being reported for the first time in this paper [7].

2.1.1 Conservation Laws

The two conserved quantities that (2.1) possess are given by [7]

$$M = \int_{-\infty}^{\infty} q dx = \frac{A}{B}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2}{n-1}\right)}{\Gamma\left(\frac{1}{2}+\frac{2}{n-1}\right)} \tag{2.13}$$

and

$$
\begin{aligned}
E &= \int_{-\infty}^{\infty} \left(q^2 + cq_{xx}^2\right) dx \\
&= \frac{A^2\left\{(n-1)^2(n+7)(3n+5) + 256(n+2)cB^4\right\}}{B(n-1)^2(n+7)(3n+5)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{4}{n-1}\right)}{\Gamma\left(\frac{1}{2}+\frac{4}{n-1}\right)}
\end{aligned}
\tag{2.14}
$$

which represents the momentum and energy of the shallow water wave respectively. The conserved quantities are computed by using the 1-soliton solution that is given by (2.3).

2.2 Singular Soliton

The singular solitary wave solution of the R-KdV equation will now be derived in this section using the ansatz method [18]. It is the same method that was employed to derive the solitary wave solution earlier in 2011 [7]. The starting point is the ansatz that is given by

$$q(x,t) = A \operatorname{csch}^p \tau \tag{2.15}$$

where

$$\tau = B(x - vt) \tag{2.16}$$

and

$$p > 0 \tag{2.17}$$

for solitons to exist. Here, in (2.15) and (2.16), A and B are free parameters, while v is the velocity of the wave. The unknown exponent p will be determined in terms of n during the process of the derivation of the soliton solution to (2.1). Substituting

(2.15) into (2.1), leads to

$$
\begin{aligned}
& \left(v-a-bp^2B^2+cvp^4B^4\right)\operatorname{csch}^p\tau \\
+\quad & B^2(p+1)(p+2)\left\{2cvB^2\left(p^2+2p+2\right)-b\right\}\operatorname{csch}^{p+2}\tau \\
+\quad & cvB^4(p+1)(p+2)(p+3)(p+4)\operatorname{csch}^{p+4}\tau-kA^{n-1}n\operatorname{csch}^{np}\tau=0.
\end{aligned}
\tag{2.18}
$$

Now, from (2.18), by the aid of balancing principle, equating the exponents pn and $p+4$ implies

$$pn=p+4 \tag{2.19}$$

so that

$$p=\frac{4}{n-1} \tag{2.20}$$

which shows that (2.4) must hold in order for solitons to exist.

Again from (2.18) the linearly independent functions are $\operatorname{csch}^{p+j}\tau$ for $j=0,2,4$. Hence, setting their respective coefficients to zero gives the same value of the soliton velocity as in (2.6) and (2.7). The relation between the soliton parameters A and B is the same as in (2.8). Then the free parameters are the same as given by (2.10) and (2.11) with their respective constraint conditions. Thus, finally, the singular 1-soliton solution for (2.1) is given by

$$q(x,t)=A\operatorname{csch}^{\frac{4}{n-1}}\left[B(x-vt)\right] \tag{2.21}$$

Chapter 3

SOLITON PERTURBATION THEORY

3.1 Modified Conservation Laws

In presence of perturbation terms, the R-KdV equation is given by [1, 7]

$$q_t + aq_x + bq_{xxx} + cq_{xxxxt} + k\left(q^n\right)_x = \epsilon R \tag{3.1}$$

where ϵ represents the perturbation parameter and R represents the perturbation terms. When these perturbation terms are turned on, the modified conservation laws are given by

$$\frac{dM}{dt} = \epsilon \int_{-\infty}^{\infty} R dx \tag{3.2}$$

$$\frac{dE}{dt} = 2\epsilon \int_{-\infty}^{\infty} qR dx \tag{3.3}$$

which respectively represent the adiabatic variation of the soliton momentum and the soliton energy. The slow change in the soliton velocity is given by

$$v = \frac{a(n-1)^4 + 16bB^2(n-1)^2}{(n-1)^4 + 256cB^4} + \frac{\epsilon}{M}\int_{-\infty}^{\infty} xR dx \tag{3.4}$$

or

$$v = \frac{b(n-1)^2}{4cB^2\left(n^2+2n+5\right)} + \frac{\epsilon}{M}\int_{-\infty}^{\infty} xR dx \tag{3.5}$$

3.2 Adiabatic Parameter Dynamics

In this paper. the specific perturbation terms that will be taken into consideration are given by [1, 9]

$$
\begin{aligned}
R &= \alpha q + \beta q_{xx} + \gamma q_x q_{xx} + \delta q^m q_x + \lambda q q_{xxx} + \nu q q_x q_{xx} \\
&+ \sigma q_x^3 + \xi q_x q_{xxxx} + \eta q_{xx} q_{xxx} + \rho q_{xxxx} + \psi q_{xxxxx} + \kappa q q_{xxxxx}
\end{aligned}
\tag{3.6}
$$

So, the perturbed R-KdV equation that is going to be considered in this paper is

$$
\begin{aligned}
& q_t + a q_x + b q_{xxx} + c q_{xxxxt} + k\left(q^n\right)_x \\
= \; & \epsilon \left(\alpha q + \beta q_{xx} + \gamma q_x q_{xx} + \delta q^m q_x + \lambda q q_{xxx} + \nu q q_x q_{xx}\right. \\
+ \; & \left.\sigma q_x^3 + \xi q_x q_{xxxx} + \eta q_{xx} q_{xxx} + \rho q_{xxxx} + \psi q_{xxxxx} + \kappa q q_{xxxxx}\right)
\end{aligned}
\tag{3.7}
$$

Amongst these perturbation, α represents the shoaling coefficient [4] and β is coefficient of dispersion [4]. The coefficient of δ is the higher order nonlinear dispersion with $1 \leq m \leq 4$ [1, 9] while the coefficient of ψ represents the fifth order spatial dispersion [1, 9]. The coefficient of ρ is due to higher order stabilizing term [1, 4, 9]. The remaining coefficients appear in the context of Whitham hierarchy of shallow water wave equation [1, 9].

In presence of these perturbation terms, the adiabatic change of the momentum is given by [1, 7]

$$\frac{dM}{dt} = \frac{\epsilon\alpha A}{B}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2}{n-1}\right)}{\Gamma\left(\frac{1}{2}+\frac{2}{n-1}\right)} = \epsilon\alpha M \tag{3.8}$$

which shows that

$$M(t) = M_0 e^{\epsilon\alpha t} \tag{3.9}$$

where M_0 is the initial momentum of the soliton. Hence, in the limiting case

$$\lim_{t\to\infty} M(t) = 0 \tag{3.10}$$

for $\alpha < 0$. This shows that the momentum adiabatically dissipates with time, in presence of shoaling since this is a dissipative perturbation term. The adiabatic change of the energy is given by [1, 9]

$$\begin{aligned}\frac{dE}{dt} &= 2\epsilon A^2\left\{\frac{\alpha}{B} - \frac{16\beta B}{(n-1)(n+7)} + \frac{256\rho B^3(n+2)}{(n-1)^2(n+7)(3n+5)}\right\}\times \\ &\times \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{4}{n-1}\right)}{\Gamma\left(\frac{1}{2}+\frac{4}{n-1}\right)}\end{aligned} \tag{3.11}$$

This means that when $dE/dt = 0$, the solitons will travel with constant energy for a stable fixed value of the width given by

$$\bar{B} = \left[\frac{\beta(n-1)(3n+5) \pm (n-1)\sqrt{(3n+5)D}}{32\rho(n+2)}\right]^{\frac{1}{2}} \tag{3.12}$$

where the discriminant D is given by

$$D = \beta^2(3n+5) - 4\alpha\rho(n+2)(n+7) \tag{3.13}$$

whenever

$$\beta^2(3n+5) > 4\alpha\rho(n+2)\,(n+7) \tag{3.14}$$

and by

$$D = 4\alpha\rho(n+2)\,(n+7) - \beta^2(3n+5) \tag{3.15}$$

for

$$\beta^2(3n+5) < 4\alpha\rho(n+2)\,(n+7) \tag{3.16}$$

If, however, $D = 0$, namely

$$\beta^2(3n+5) = 4\alpha\rho(n+2)\,(n+7) \tag{3.17}$$

the fixed value of the soliton width is given by

$$\bar{B} = \left[\frac{\beta(n-1)(3n+5)}{32\rho\,(n+2)}\right]^{\frac{1}{2}} \tag{3.18}$$

in which case the constraint condition

$$\beta\rho > 0 \tag{3.19}$$

is valid. Also, the corresponding fixed value of the amplitude, in this case by virtue of (2.5), is

$$\bar{A} = \left[\frac{(n+1)(n+3)(3n+1)b\left\{\beta(3n+5) + \sqrt{(3n+5)D}\right\}}{16k\rho\,(n^2+2n+5)\,(n-1)(n+2)} \right]^{\frac{1}{n-1}} \tag{3.20}$$

where the same analysis as in (3.13)-(3.16) holds. If however, the discriminant vanishes by virtue of (3.17), the fixed value of the amplitude is then given by

$$\bar{A} = \left[\frac{(n+1)(n+3)(3n+1)(3n+5)b\beta}{16k\rho\,(n^2+2n+5)\,(n-1)(n+2)} \right]^{\frac{1}{n-1}} \tag{3.21}$$

in which case the constraint condition reduces to

$$bk\rho\beta > 0 \tag{3.22}$$

for odd values of n.

For these specific perturbation terms, given by (3.6), the slow change in the velocity of the soliton is

$$\begin{aligned} v &= \frac{a(n-1)^4 + 16bB^2(n-1)^2}{(n-1)^4 + 256cB^4} \\ &- \epsilon B \left[\frac{\delta A^m}{m+1} \frac{\Gamma\left(\frac{2m+2}{n-1}\right)}{\Gamma\left(\frac{2m+2}{n-1}+\frac{1}{2}\right)} + \frac{16A^2B^2\left\{\nu(n-1)^2+\sigma(n+17)\right\}}{3(n-1)(n+5)(n+11)} \frac{\Gamma\left(\frac{6}{n-1}\right)}{\Gamma\left(\frac{6}{n-1}+\frac{1}{2}\right)} \right. \\ &+ \left. \frac{8AB^2\left\{(n-1)(3n+5)(\gamma-3\lambda) - 16(n+2)(3\xi-\eta-5\kappa)B^2\right\}}{(n-1)^2(n+7)(3n+5)} \frac{\Gamma\left(\frac{4}{n-1}\right)}{\Gamma\left(\frac{4}{n-1}+\frac{1}{2}\right)} \right] \times \\ &\times \frac{\Gamma\left(\frac{2}{n-1}+\frac{1}{2}\right)}{\Gamma\left(\frac{2}{n-1}\right)} \end{aligned} \tag{3.23}$$

or

$$
\begin{aligned}
v &= \frac{b(n-1)^2}{4cB^2\left(n^2+2n+5\right)} \\
&- \epsilon B\left[\frac{\delta A^m}{m+1}\frac{\Gamma\left(\frac{2m+2}{n-1}\right)}{\Gamma\left(\frac{2m+2}{n-1}+\frac{1}{2}\right)} + \frac{16A^2B^2\left\{\nu(n-1)^2+\sigma(n+17)\right\}}{3(n-1)(n+5)(n+11)}\frac{\Gamma\left(\frac{6}{n-1}\right)}{\Gamma\left(\frac{6}{n-1}+\frac{1}{2}\right)}\right. \\
&+ \left.\frac{8AB^2\left\{(n-1)(3n+5)(\gamma-3\lambda)-16(n+2)(3\xi-\eta-5\kappa)B^2\right\}}{(n-1)^2(n+7)(3n+5)}\frac{\Gamma\left(\frac{4}{n-1}\right)}{\Gamma\left(\frac{4}{n-1}+\frac{1}{2}\right)}\right]\times \\
&\times \frac{\Gamma\left(\frac{2}{n-1}+\frac{1}{2}\right)}{\Gamma\left(\frac{2}{n-1}\right)} \qquad (3.24)
\end{aligned}
$$

by virtue of (3.4) and (3.5).

Chapter 4
EXACT SOLUTION (ANSATZ METHOD)

The exact 1-soliton solution to the perturbed R-KdV equation for strong perturbation terms will be now obtained by the aid of the ansatz method. The perturbed R-KdV equation that is going to be studied in this section is given by [1, 9]

$$\begin{aligned} & q_t + aq_x + bq_{xxx} + cq_{xxxxt} + k\left(q^n\right)_x \\ = \; & \gamma q_x q_{xx} + \lambda q q_{xxx} + \nu q q_x q_{xx} + \sigma q_x^3 \\ + \; & \xi q_x q_{xxxx} + \eta q_{xx} q_{xxx} + \psi q_{xxxxx} + \kappa q q_{xxxxx} \end{aligned} \tag{4.1}$$

where certain perturbation terms due to shoaling, stabilization and second order dispersion are removed. These removed terms are dissipative, as seen earlier, and do not permit integrability of the perturbed R-KdV equation. Equation (4.1) will now be split into the following two subsections for solitary waves and singular solitons.

4.1 Solitary Waves

For solitary wave solution to (4.1), the starting hypothesis is given by

$$q(x, t) = A\,\mathrm{sech}^p\left[B(x - vt)\right] \tag{4.2}$$

where p is the unknown exponent that will be determined by the aid of the balancing principle. Then, by the ansatz method, the amplitude (A), width (B) and velocity

(v) will be determined. Substituting (4.2) into (4.1) leads to

$$
\begin{aligned}
& \left(v - a - bp^2B^2 + cvp^4B^4 + \psi p^4B^4\right) \operatorname{sech}^p\tau \\
+ \quad & (p+1)(p+2)\left[b - 2\left(p^2+2p+2\right)(cv+\psi)B^2\right] B^2 \operatorname{sech}^{p+2}\tau \\
+ \quad & (p+1)(p+2)(p+3)(p+4)(cv+\psi)B^4 \operatorname{sech}^{p+4}\tau - knA^{n-1}\operatorname{sech}^{np}\tau \\
+ \quad & \left(\gamma + \lambda + \xi p^2B^2 + \eta p^2B^2 + \kappa p^2B^2\right) p^2AB^2 \operatorname{sech}^{2p}\tau \\
- \quad & (p+1)AB^2\left[\gamma p + (p+2)\lambda + 2p^2(p+1)\eta B^2\right. \\
+ \quad & \left. 2\left(p^2+2p+2\right)\{\xi p + (p+2)\kappa\} B^2\right] \operatorname{sech}^{2p+2}\tau \\
+ \quad & (p+1)(p+2)AB^4\{p(p+3)\xi + p(p+1)\eta + (p+3)(p+4)\kappa\} \operatorname{sech}^{2p+4}\tau \\
+ \quad & p^2A^2B^2(\sigma+\nu)\operatorname{sech}^{3p}\tau - pA^2B^2\{p\sigma + (p+1)\nu\}\operatorname{sech}^{3p+2}\tau = 0
\end{aligned}
\tag{4.3}
$$

Then, from (4.3) equating the exponents np and $p+4$, by the aid of balancing principle, leads to the same situation as in (2.19) and (2.20). Again, as before, from the linearly independent functions, the amplitude-width relation is still given by (2.5) while the velocity of the soliton is given by

$$v = \frac{a(n-1)^4 + 16bB^2(n-1)^2 - 256\psi B^4}{(n-1)^4 + 256cB^4} \tag{4.4}$$

or

$$v = \frac{b(n-1)^2 - 4\psi\left(n^2+2n+5\right)B^2}{4cB^2\left(n^2+2n+5\right)} \tag{4.5}$$

Then equating these two expression for velocity leads to the relation

$$b(n-1)^4 = 4B^2\left[\left(n^2+2n+5\right)\left\{(n-1)^2(ac+\psi) + 16bcB^2\right\} - 64bcB^2\right] \tag{4.6}$$

and solving this bi-quadratic equation for the width B leads to

$$B = \frac{n-1}{n+1}\left[\frac{-(ac+\psi)\left(n^2+2n+5\right)+\sqrt{(ac+\psi)^2\left(n^2+2n+5\right)^2+16b^2c(n+1)^2}}{32bc}\right]^{\frac{1}{2}} \tag{4.7}$$

Substituting (4.7) into (2.5) leads to

$$\begin{aligned} A \;=\; & \left[-(ac+\psi)\left(n^2+2n+5\right)+\sqrt{(ac+\psi)^2\left(n^2+2n+5\right)^2+16b^2c(n+1)^2}\right]^{\frac{1}{n-1}} \times \\ & \times \left[\frac{(n+3)(3n+1)}{16ck(n+1)\left(n^2+2n+5\right)}\right]^{\frac{1}{n-1}} \end{aligned} \tag{4.8}$$

It needs to be noted that, when $\psi = 0$, the relations (4.6)-(4.8) collapse to (2.8), (2.10) and (2.11) respectively.

The additional linearly independent functions from (4.3) leads to a set of constraint relations given by

$$\sigma + \nu = 0 \tag{4.9}$$

$$4\sigma + (n+3)\nu = 0 \tag{4.10}$$

This linear system of equations given by (4.9) and (4.10) implies the unique solution

$$\begin{bmatrix} \sigma \\ \nu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.11}$$

since the determinant of the coefficient matrix is given by

$$D = n - 1 \tag{4.12}$$

and consequently $D \neq 0$ since $n \neq 1$ as indicated in (2.2).

Furthermore, there is another set of constraint relations that must remain valid for the integrability of (4.1) by the ansatz method. These are given by

$$\begin{aligned} & (n-1)^2 \{2\gamma + (n+1)\lambda\} \\ + \; & 2\left[8(n+3)\eta + \left(n^2+2n+5\right)\{(n+3)\xi + 2(n+1)\kappa\}\right] B^2 = 0 \end{aligned} \tag{4.13}$$

$$(3n+1)\xi + (n+3)\eta + n(3n+1)\kappa = 0 \tag{4.14}$$

$$(n-1)^2(\gamma+\lambda) + 16(\xi+\eta+\kappa)B^2 = 0 \tag{4.15}$$

This set of linear equations, given by (4.13)-(4.15) gives the solution

$$\begin{bmatrix} \xi \\ \eta \\ \kappa \\ \gamma \\ \lambda \end{bmatrix} = \begin{bmatrix} -\frac{(n-1)^2\left(n^2-4n-9\right)}{8B^2(3n^3+17n^2+29n-1)} \\ -\frac{(n-1)(3n+1)\left(n^3+4n^2+9n-6\right)}{16B^2(3n^3+17n^2+29n-1)} \\ \frac{(n-1)\left(n^3+9n^2+11n+11\right)}{16B^2(3n^3+17n^2+29n-1)} \\ 1 \\ 0 \end{bmatrix} \gamma + \begin{bmatrix} -\frac{(n-1)^2\left(n^2+8n+3\right)}{8B^2(3n^3+17n^2+29n-1)} \\ -\frac{(n-1)(n+2)(3n+1)\left(n^2+1\right)}{16B^2(3n^3+17n^2+29n-1)} \\ \frac{(n-1)\left(n^3+9n^2+27n-5\right)}{16B^2(3n^3+17n^2+29n-1)} \\ 0 \\ 1 \end{bmatrix} \lambda \tag{4.16}$$

Thus, the system of linear equations (4.13)-(4.15) for the perturbation coefficients lead to the conclusion that the five perturbation coefficients, ξ, η, κ, γ and λ can be solved in terms of two linearly independent parameters, namely γ and λ only. This is because the matrix coefficient of the system of linear equations as indicated by (4.13)-(4.15) has rank 2. Hence, their solution matrix given by (4.16) has dimension 2. Therefore, in principle, equation (4.1) can be solved by the ansatz method provided the perturbation coefficients satisfy relations (4.11) and (4.16).

4.2 Singular Solitons

For singular solitons of the perturbed R-KdV equation given by (4.1), the starting hypothesis is still taken to be (2.15). Then substituting (2.15) into (4.1) gives

$$
\begin{aligned}
& \left(v - a - bp^2B^2 + cvp^4B^4 + \psi p^4B^4\right) \operatorname{csch}^p\tau \\
- \; & (p+1)(p+2)\left[b - 2\left(p^2+2p+2\right)(cv+\psi)B^2\right] B^2 \operatorname{csch}^{p+2}\tau \\
+ \; & (p+1)(p+2)(p+3)(p+4)(cv+\psi)B^4 \operatorname{csch}^{p+4}\tau - knA^{n-1} \operatorname{csch}^{np}\tau \\
+ \; & \left(\gamma + \lambda + \xi p^2B^2 + \eta p^2B^2 + \kappa p^2B^2\right) p^2AB^2 \operatorname{csch}^{2p}\tau \\
+ \; & (p+1)AB^2\left[\gamma p + (p+2)\lambda + 2p^2(p+1)\eta B^2\right. \\
+ \; & \left. 2\left(p^2+2p+2\right)\{\xi p + (p+2)\kappa\} B^2\right] \operatorname{csch}^{2p+2}\tau \\
+ \; & (p+1)(p+2)AB^4\{p(p+3)\xi + p(p+1)\eta + (p+3)(p+4)\kappa\} \operatorname{csch}^{2p+4}\tau \\
+ \; & p^2A^2B^2(\sigma+\nu) \operatorname{csch}^{3p}\tau + pA^2B^2\{p\sigma + (p+1)\nu\} \operatorname{csch}^{3p+2}\tau = 0 \qquad (4.17)
\end{aligned}
$$

Thus, following the same procedure as in the previous sub-section for solitary waves, this analysis leads to exactly the same results as given by (4.9)-(4.16). The free parameters A and B are still given by (4.8) and (4.7) respectively.

Chapter 5

SEMI-INVERSE VARIATIONAL PRINCIPLE

The previous section carried out the integration of the perturbed R-KdV equation by the ansatz method. In this section, there is another analytical approach that will be adopted to integrate the perturbed R-KdV equation. This is the semi-inverse variational principle (SVP) [8, 10, 12]. This technique, however, does not give an exact solution, unlike the ansatz method as discussed in the previous section, although this is a purely analytical approach to solve the equation. Therefore, there are fewer perturbation terms that can be incorporated in order to be able to apply the SVP. Thus, the perturbed R-KdV equation that will be considered in this section is given by [8, 10, 12]

$$q_t + aq_x + bq_{xxx} + cq_{xxxxt} + k\left(q^n\right)_x = \delta q^m q_x + \psi q_{xxxxx} \tag{5.1}$$

As seen from (5.1). it is only the nonlinear dispersion and the fifth order dispersion terms that permit integrability of the perturbed R-KdV equation. The first step is to consider the traveling wave hypothesis that is taken to be

$$q(x,t) = g(x - vt) = g(s) \tag{5.2}$$

where

$$s = x - vt \tag{5.3}$$

Substituting this traveling wave assumption into (5.1) and integrating once while taking the integration constant to be zero gives

$$(v-a)g - bg'' + (cv+\psi)g'''' - kg^n + \frac{\delta}{m+1}g^{m+1} = 0 \tag{5.4}$$

where the notations $g'' = d^2g/ds^2$ and $g'''' = d^4g/ds^4$ are adopted. Now, multiplying both sides of (5.4) by $g' = dg/ds$ and integrating leads to

$$\begin{aligned} &(v-a)g^2 - b\left(g'\right)^2 + (cv+\psi)\left\{2g'g''' - \left(g''\right)^2\right\} \\ - \quad & 2k\frac{g^{n+1}}{n+1} + 2\delta\frac{g^{m+2}}{(m+1)(m+2)} = K \end{aligned} \tag{5.5}$$

where K is the integration constant. The stationary integral is then defined as

$$\begin{aligned} J &= \int_{-\infty}^{\infty} K ds \\ &= \int_{-\infty}^{\infty} \left[(v-a)g^2 - b\left(g'\right)^2 + (cv+\psi)\left\{2g'g''' - \left(g''\right)^2\right\}\right. \\ &- \left. 2k\frac{g^{n+1}}{n+1} + 2\delta\frac{g^{m+2}}{(m+1)(m+2)}\right] ds \end{aligned} \tag{5.6}$$

Now, choosing

$$g(s) = A\,\mathrm{sech}^{\frac{4}{n-1}}(Bs) \tag{5.7}$$

as a solution hypothesis for (5.1), where A and B are still the amplitude and inverse width of the soliton, the stationary integral J reduces to

$$
\begin{aligned}
J &= \frac{2\delta A^{2m+2}}{(m+1)(m+2)B} \frac{\Gamma\left(\frac{2m+4}{n-1}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2m+4}{n-1}+\frac{1}{2}\right)} \\
&+ \left[\frac{(v-a)A^2}{B} - \frac{16bA^2B}{(n-1)(n+7)} - \frac{768\,(n+2)\,(cv+\psi)A^2B^3}{(n-1)^2(n+7)(3n+5)}\right. \\
&- \left.\frac{32(n+3)KA^{n+1}}{(n+1)(n+7)(3n+5)B}\right] \frac{\Gamma\left(\frac{4}{n-1}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{4}{n-1}+\frac{1}{2}\right)}. \qquad (5.8)
\end{aligned}
$$

Then, the SVP states that the amplitude (A) and the width (B) of the perturbed R-KdV equation (5.1) is given by the solution of the coupled system of equations [8, 10, 12]

$$\frac{\partial J}{\partial A} = 0 \qquad (5.9)$$

and

$$\frac{\partial J}{\partial B} = 0 \qquad (5.10)$$

From (5.8), equations (5.9) and (5.10) are respectively given by

$$
\begin{aligned}
& v - a - \frac{16bB^2}{(n-1)(n+7)} - \frac{768(n+2)(cv+\psi)B^4}{(n-1)^2(n+7)(3n+5)} \\
&+ \frac{\delta A^m}{m+1} \frac{\Gamma\left(\frac{2m+4}{n-1}\right)\Gamma\left(\frac{4}{n-1}+\frac{1}{2}\right)}{\Gamma\left(\frac{2m+4}{n-1}+\frac{1}{2}\right)\Gamma\left(\frac{4}{n-1}\right)} - \frac{16k(n+3)A^{n-1}}{(n+7)(3n+5)} = 0 \qquad (5.11)
\end{aligned}
$$

and

$$v - a + \frac{16bB^2}{(n-1)(n+7)} + \frac{2304(n+2)(cv+\psi)B^4}{(n-1)^2(n+7)(3n+5)}$$
$$+ \frac{2\delta A^m}{(m+1)(m+2)} \frac{\Gamma\left(\frac{2m+4}{n-1}\right)\Gamma\left(\frac{4}{n-1}+\frac{1}{2}\right)}{\Gamma\left(\frac{2m+4}{n-1}+\frac{1}{2}\right)\Gamma\left(\frac{4}{n-1}\right)} - \frac{32k(n+3)A^{n-1}}{(n+1)(n+7)(3n+5)} = 0 \tag{5.12}$$

after simplification. Subtracting (5.11) from (5.12) leads to the bi-quadratic equation for the width of the soliton as

$$256NB^4 + 32(n-1)(n+7)bB^2 - M(n-1)^2(n+7)^2 = 0 \tag{5.13}$$

where

$$M = \frac{m\delta A^m}{(m+1)(m+2)} \frac{\Gamma\left(\frac{2m+4}{n-1}\right)\Gamma\left(\frac{4}{n-1}+\frac{1}{2}\right)}{\Gamma\left(\frac{2m+4}{n-1}+\frac{1}{2}\right)\Gamma\left(\frac{4}{n-1}\right)} - \frac{16k(n-1)(n+3)A^{n-1}}{(n+1)(n+7)(3n+5)} \tag{5.14}$$

and

$$N = \frac{12(n+2)(n+7)(cv+\psi)}{3n+5} \tag{5.15}$$

The solution of (5.13) is then

$$B = \frac{1}{4}\left[\frac{(n-1)(n+7)\left\{-b+\sqrt{b^2+NM}\right\}}{N}\right]^{\frac{1}{2}} \tag{5.16}$$

for

$$MN > 0 \tag{5.17}$$

and

$$N\left(-b+\sqrt{b^2+NM}\right)>0 \tag{5.18}$$

Finally, the amplitude can be obtained from the polynomial equation (5.11) or (5.12). Also, the velocity of the soliton can also be obtained directly from either (5.11) or (5.12). Once these parameters are obtained, the soliton solution of the perturbed R-KdV equation can be written as (5.7), which is the analytical soliton solution of the perturbed R-KdV equation (5.1).

Chapter 6

CONCLUSIONS AND FUTURE WORK

This paper addresses the dynamics of dispersive solitary waves that is modeled by the R-KdV equation with power law nonlinearity in presence of perturbation terms. The 1-soliton solution is obtained by the ansatz method for solitary waves and singular solitons. Then, the soliton perturbation theory is implemented in order to describe the adiabatic dynamics of the perturbed soliton. It was observed that the momentum dissipates in presence of shoaling. while the solitons travel with a fixed value of the energy when the amplitude and the width of the soliton get locked to a definitive value. Finally, the integration of the perturbed R-KdV equation is carried out by the aid of two integration tools. They are the ansatz method as well as the SVP. These results lead to a set of constraint conditions that must hold in order for the soliton dynamics to hold.

In this context, the future of this research holds pretty strong. These results will be extended further. The soliton solutions will be obtained with stochastic perturbation terms after integrating the corresponding Langevin equation. This will lead to a mean free value of the soliton parameters in presence of such perturbation terms. The quasi-stationary soliton solution will also be obtained by the aid of multiple-scale perturbation theory. and finally the quasi-particle theory of soliton-soliton interaction will be developed. These just form the tip of the iceberg.

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