# A STUDY ON THE NUMERICAL AND ANALYTICAL SOLUTIONS <br> OF COMPLEX-VARIABLE PARTIAL <br> DIFFERENTIAL EQUATIONS 

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## A DISSERTATION

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## DEDICATION

I dedicate this thesis to all those who have ever thought that math is impossible, difficult, or would never be used in the real world. To many people who have been "defeated" by mathematics, they have not been able to see the true beauty and magic that the field of mathematics has to offer, that same beauty and magic that I have had the blessing of experiencing. I hope that through my teaching and research, I can broaden the population of people who have interest in mathematics, or at the least, not be afraid of learning more about it.

## ACKNOWLEDGMENTS

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#### Abstract

In this work, we consider the analogue of a real-variable partial differential equation. In comparison to what has already been thoroughly investigated, recall the non-linear Schrodinger equation (NLSE). The NLSE, which is used in determining the wave equation for quantum particles, is a real-variable PDE with complex coefficients. Instead, we consider equations where both the function $\omega$ and its independent variable $z$ belong to the complex plane. We approach the complex problem by an intuitive approach of treating a one-complex variable differential equation as a two-real variable partial differential equation by analyzing the real and imaginary parts of both $\omega$ and $z$. We investigate thoroughly the first-order complex PDE case and prove the existence and uniqueness theorem for these types of equations. We also investigate the analytical solutions by considering the complex-variable Laplace transform, which can be thought of in parallel as a two-variable Laplace transform with in $\mathbb{R}^{2}$. Upon completion of the first-order case, we consider the higher order complex-variable PDE. We discuss both the direct way of solving higher-order equations via systems of real-variable PDE's and also via first-order systems of complex-variable PDE's, in which we implement the methods of the previous topics. As a direct consequence of the higher-order differential equation solution method, we also discuss an alternative method of evaluating complex contour integrals via a real-variable partial differential equation evaluation. To conclude, we consider the time-dependent complex variable PDE analogues of the advection and wave equations, we briefly discuss multi-complex variable PDE's and methods that we plan to investigate in the near future.


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## Chapter 1 <br> LITERATURE REVIEW

### 1.1 Functions of a Complex Variable

In this first chapter, we lay down the foundations of the topics that we plan to discuss, define some of the commonly used terminology, and discuss some of the main results that we use throughout this thesis. Some of the things mentioned here we may not use, but it will help provide some contexts of what and why we are doing what we are doing, and what we plan on doing with it.

### 1.1.1 Complex Numbers

We will be primarily considering the set of complex numbers, denoted $\mathbb{C}$. The set of complex numbers is defined by:

$$
\begin{equation*}
\mathbb{C}:=\{a+b i: a, b \in \mathbb{R}\} \tag{1.1}
\end{equation*}
$$

where $\mathbb{R}$ denotes the set of real numbers and $i:=\sqrt{-1}$. We define the operations of two complex numbers $z_{1}:=a_{1}+b_{1} i, z_{2}:=a_{2}+b_{2} i$ in the standard way:

$$
\begin{align*}
z_{1} \pm z_{2} & =\left(a_{1} \pm a_{2}\right)+\left(b_{1} \pm b_{2}\right) i \\
z_{1} \cdot z_{2} & =\left(a_{1} a_{2}-b_{1} b_{2}\right)+\left(a_{1} b_{2}+a_{2} b_{1}\right) i  \tag{1.2}\\
z_{1} \div z_{2} & =\frac{a_{1} a_{2}+b_{1} b_{2}}{a_{2}^{2}+b_{2}^{2}}+\frac{a_{2} b_{1}-a_{1} b_{2}}{a_{2}^{2}+b_{2}^{2}} i
\end{align*}
$$

The conjugate of a complex number $z=a+b i$, denoted $\bar{z}$ is defined by:

$$
\begin{equation*}
\bar{z}:=a-b i \tag{1.3}
\end{equation*}
$$

which satisfies the following property:

$$
\begin{equation*}
z \cdot \bar{z}=a^{2}+b^{2} . \tag{1.4}
\end{equation*}
$$

The number $a$ will be referred to as the 'real part' of $z$, and $b$ the 'imaginary part' of $z$, each of which will be respectively denoted by:

$$
\begin{equation*}
\operatorname{Re}(z):=a \quad \operatorname{Im}(z):=b \tag{1.5}
\end{equation*}
$$

The 'modulus' of the complex number $z$ is given by:

$$
\begin{equation*}
|z| \equiv r:=\sqrt{a^{2}+b^{2}} \tag{1.6}
\end{equation*}
$$

and the 'argument' is given by:

$$
\begin{equation*}
\arg (z) \equiv \theta:=\arctan \left(\frac{b}{a}\right) \tag{1.7}
\end{equation*}
$$

These concepts can be represented graphically in Figure 1.1. From Euler's formula, we can


Figure 1.1: Graphical Representation of Complex Numbers
represent any complex number $z$ in its polar form by:

$$
\begin{equation*}
z=r e^{i \theta}=r(\cos \theta+i \sin \theta) \tag{1.8}
\end{equation*}
$$

### 1.1.2 Complex-Variable Functions

We now consider a function $f: \mathbb{C} \rightarrow \mathbb{C}$. For real-variable functions, we graphically represent the input value on the horizontal axis and the output variable on the vertical axis. Since the input value for a complex-variable function is represented as a two-dimensional quantity (namely the real and imaginary part), the output is also a two-dimensional quantity, resulting in a four-dimensional function. One method of visualizing a complex function is analyzing the real and imaginary part of a complex function. For example, we can rewrite a complex number as $z:=x+i y$. Therefore the image of $z$ under $f$ must be related to the variables $x, y$, where a portion of it is real and the other imaginary. We denote the $\operatorname{Re}(f(z)):=u(x, y)$ and the $\operatorname{Im}(f(z)):=v(x, y)$, giving us:

$$
\begin{equation*}
f(z)=f(x+i y)=u(x, y)+v(x, y) i . \tag{1.9}
\end{equation*}
$$

Being able to obtain the real and imaginary part of a function will be crucial in the next chapter to compare our numerical results to the exact results, so we will devote this section on how to find the real and imaginary parts of the basic functions and give an example on the composition of such functions.

As a basic example, consider $f(z)=z^{2}$. Replacing $z$ with $x+i y$, expanding and rearranging, we can obtain:

$$
\begin{equation*}
f(z)=\left(x^{2}-y^{2}\right)+(2 x y) i . \tag{1.10}
\end{equation*}
$$

From here, we can define the real part $u(x, y):=x^{2}-y^{2}$ and the imaginary part $v(x, y):=$ $2 x y$. It is beneficial to generalize this to the $n^{\text {th }}$ degree polynomial $\varphi_{1}(z)=z^{n}$, where $n \in \mathbb{N}$. Applying the binomial theorem to $x+i y$, we have:

$$
\begin{align*}
\varphi_{1}(z) & =(x+i y)^{n} \\
& =\sum_{k=0}^{n}\binom{n}{k} x^{n-k}(i y)^{k} \tag{1.11}
\end{align*}
$$

Since the powers of $i$ form a cyclic group of order 4, and also the following identities hold:

$$
\begin{array}{ll}
i^{4 k+0}=1 & i^{4 k+2}=-1  \tag{1.12}\\
i^{4 k+1}=i & i^{4 k+3}=-i
\end{array}
$$

where $k \in \mathbb{Z}$, then we can reorder (Eq 1.11) in terms of its real and imaginary parts as:

$$
\begin{align*}
\varphi_{1}(z)= & \left(\sum_{k=0}^{n}\binom{n}{2 k}(-1)^{k} x^{n-2 k} y^{2 k}\right)  \tag{1.13}\\
& +\left(\sum_{k=0}^{n}\binom{n}{2 k+1}(-1)^{k} x^{n-2 k-1} y^{2 k+1}\right) i .
\end{align*}
$$

For rational functions, consider $\varphi_{2}(z)=\frac{1}{z}$. The representation of this comes as a direct result of complex division:

$$
\begin{align*}
\varphi_{2}(z) & =\frac{1}{z}  \tag{1.14}\\
& =\frac{x}{x^{2}-y^{2}}-\frac{y}{x^{2}-y^{2}} i
\end{align*}
$$

For the general radical function $\varphi_{3}(z)=\sqrt[n]{z}$, we begin by writing $z$ in its polar form $r e^{i \theta}$ (see Eq 1.6 and 1.7 for definitions of $r$ and $\theta$ ), apply the properties of radicals and Euler's formula ( Eq 1.8 ) to obtain the desired form:

$$
\begin{align*}
\varphi_{3}(z) & =\sqrt[n]{z} \\
& =\sqrt[n]{r} \cos \left(\frac{\theta}{n}\right)+\sqrt[n]{r} \sin \left(\frac{\theta}{n}\right) i \tag{1.15}
\end{align*}
$$

Note here that $\sqrt[n]{r}$ is multi-valued with $n$ different values; hence one may choose to only focus on the principal value (branch) of the function. Using Euler's identity directly for the exponential function:

$$
\begin{align*}
\varphi_{4}(z) & =e^{z}  \tag{1.16}\\
& =e^{x} \cdot \cos (y)+e^{x} \cdot \sin (y) i
\end{align*}
$$

one may implement the following identities to obtain representations for $\sin z$ and $\cos z$ :

$$
\begin{equation*}
\sin z=\frac{e^{i z}-e^{-i z}}{2 i}, \quad \cos z=\frac{e^{i z}+e^{-i z}}{2} \tag{1.17}
\end{equation*}
$$

As an alternative, one can use the definitions of the hyperbolic trigonometric functions:

$$
\begin{equation*}
\sinh z=\frac{e^{z}-e^{-z}}{2}, \quad \cosh z=\frac{e^{z}+e^{-z}}{2} \tag{1.18}
\end{equation*}
$$

together with the easily-verifiable identities:

$$
\begin{equation*}
\sin (i z)=i \sinh (z) \quad \cos (i z)=\cosh (z) \tag{1.19}
\end{equation*}
$$

and applying the sum identity for sine (and similarly for cosine) to obtain:

$$
\begin{align*}
\varphi_{5}(z) & =\sin (z)  \tag{1.20}\\
& =(\sin x \cosh y)+(\sinh y \cos x) i
\end{align*}
$$

For the logarithm $\ln (z)$, we can again replace $z$ with its polar representation to obtain:

$$
\begin{align*}
\varphi_{6}(z) & =\ln (z)  \tag{1.21}\\
& =\ln \sqrt{x^{2}+y^{2}}+\arctan \left(\frac{y}{x}\right) i, x \neq y \neq 0 .
\end{align*}
$$

For the tangent function, we can begin with the quotient identity and use the real and imaginary representation for sine and cosine to obtain:

$$
\begin{align*}
\varphi_{7}(z)=\tan (z) & =\frac{\sin (z)}{\cos (z)}  \tag{1.22}\\
& =\frac{(\sin x \cosh y)+(\sinh y \cos x) i}{(\cos x \cosh y)-(\sin x \sinh y) i}
\end{align*}
$$

The denominator of (Eq 1.22) is, for any pair $(x, y)$ a complex number, so we can multiply top and bottom by its conjugate and use both the circular and hyperbolic Pythagorean
identities to simplify in order to obtain the desired representation:

$$
\begin{align*}
\varphi_{7}(z) & =\tan (z) \\
& =\frac{(\sin x \cos x)+(\sinh y \cosh y) i}{\cos ^{2} x \cosh ^{2} y+\sin ^{2} x \sinh ^{2} y} . \tag{1.23}
\end{align*}
$$

To find the real and imaginary parts of the function $\omega:=\arctan (z)$, we can first write this as $\tan (\omega)=z$. Using both parts of (Eq 1.17) to rewrite tangent, we can obtain through algebraic manipulation:

$$
\begin{equation*}
z=\frac{e^{i \omega}-\frac{1}{e^{i \omega}}}{\left(e^{i \omega}+\frac{1}{e^{i \omega}}\right) i} \quad \rightarrow \quad e^{2 \omega i}=\frac{1+i z}{1-i z} \tag{1.24}
\end{equation*}
$$

Solving this for $\omega$ gives us the identity:

$$
\begin{equation*}
\arctan (z)=\frac{i}{2} \cdot \ln \left(\frac{1-i z}{1+i z}\right) \tag{1.25}
\end{equation*}
$$

Now that we have represented our function in terms of a logarithm, we can implement (Eq 1.21). First note that the interior of the logarithm is equal to $\frac{(1+y)-(x) i}{(1-y)+(x) i}$, so we can multiply top and bottom by the denominator conjugate to express the interior as an $(x, y)$-dependent complex number to apply (Eq 1.21) to. This gives us:

$$
\begin{equation*}
\arctan (z)=\frac{i}{2} \cdot \ln \left(\frac{1-x^{2}-y^{2}}{(1-y)^{2}+x^{2}}-\frac{2 x}{(1-y)^{2}+x^{2}} i\right):=\frac{i}{2} \cdot \ln (\alpha+\beta i) \tag{1.26}
\end{equation*}
$$

where $\alpha, \beta$ are the real- and imaginary- parts of the interior of the logarithm, which are of course functions of $x, y$. We then have that the modulus and argument of the logarithmic interior is given by:

$$
\begin{equation*}
r=\sqrt{\alpha^{2}+\beta^{2}}, \quad \theta=\arctan \left(\frac{\beta}{\alpha}\right) \tag{1.27}
\end{equation*}
$$

hence one we apply the properties of logarithms to $\alpha+\beta i=r e^{i \theta}$, applying (Eq 1.21) and
rearranging, we obtain our relationship:

$$
\begin{align*}
\varphi_{8}(z) & =\arctan (z) \\
& =\frac{1}{2} \arctan \left(-\frac{\beta}{\alpha}\right)+\frac{1}{4} \ln \left(\alpha^{2}+\beta^{2}\right) i . \tag{1.28}
\end{align*}
$$

As an example (which we will use in the next chapter), suppose that we want the real and imaginary part of the following constant:

$$
\begin{equation*}
c=\arctan (3-7 i)-\ln (4-7 i) . \tag{1.29}
\end{equation*}
$$

For the inverse tangent, we have that $(x, y)=(3,-7)$, hence $(\alpha, \beta)=\left(\frac{-57}{73}, \frac{-6}{73}\right)$. Since our $(\alpha, \beta)$ lies in quadrant III, we will add $\frac{\pi}{2}$ for the appropriate branch of the inverse tangent; hence we obtain:

$$
\begin{align*}
\arctan (3-7 i) & =\frac{1}{2} \arctan \left(\frac{6}{57}\right)+\frac{\pi}{2}+\frac{1}{4} \ln \left((57 / 73)^{2}+(6 / 73)^{2}\right) i  \tag{1.30}\\
& \approx 1.5184-0.1209 i
\end{align*}
$$

Similarly for the logarithmic portion, we can directly use (Eq 1.21) to obtain:

$$
\begin{align*}
\ln (4-7 i) & =\frac{1}{2} \ln \left(4^{2}+(-7)^{2}\right)+\arctan (-7 / 4) i  \tag{1.31}\\
& \approx 2.0872-1.0517 i
\end{align*}
$$

Combining these results give us the real and imaginary part of (Eq 1.29) to be:

$$
\begin{align*}
c & =\arctan (3-7 i)-\ln (4-7 i)  \tag{1.32}\\
& \approx-0.5688+0.9307 i
\end{align*}
$$

Instead of analyzing $\varphi$ in two pieces $u, v$, we can instead visualize it in one 'four-dimensional' graph. We can graph $x, y, u$ as a 3 -dimensional plot and then use a color map to describe the range of $v$; this approach is known as the Riemann surface representation[20]. For
comparison of the two, we have included the real-imaginary decomposition plots as well as the Riemann surface graph for the function $\varphi_{6}$ defined above. One may also graph $x, y, r$ as a 3 -dimensional plit and then use a color map to describe the range of $\theta$, sometimes referred to as an modulus-argument surface. For other representations that use other software, one may refer to [12].


Figure 1.2: Riemann Surface for the Logarithm


Figure 1.3: Real/Imaginary Parts for the Logarithm


Figure 1.4: Modulus-Argument Surface for the Logarithm

### 1.1.3 Differential and Integral Calculus

If for any $\epsilon>0$ there exists a $\delta>0$ such that $0<\left|z-z_{0}\right|<\delta$ implies that $\left|f(z)-\omega_{0}\right|<\epsilon$, then we say that the limit of $f$ as $z$ approaches $z_{0}$ is $w_{0}$ and write:

$$
\begin{equation*}
\lim _{z \rightarrow z_{0}} f(z)=\omega_{0} \tag{1.33}
\end{equation*}
$$

We define continuity in the traditional way by saying that $f$ is continuous at $z_{0}$ if:

$$
\begin{equation*}
\lim _{z \rightarrow z_{0}} f(z)=f\left(z_{0}\right) \tag{1.34}
\end{equation*}
$$

The derivative of $f$ is given by:

$$
\begin{equation*}
\frac{d}{d z} f(z) \equiv f^{\prime}(z)=\lim _{\Delta z \rightarrow 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} \tag{1.35}
\end{equation*}
$$

Recall that for the derivative to exist, the aforementioned limit must be unique regardless of the path that you take for $\Delta z \rightarrow 0$. If the derivative of $f$ exists at all $z \in \Omega \subset \mathbb{C}$, then we say that $f$ is 'analytic' on $\Omega$. If $\Omega=\mathbb{C}$, then we refer to $f$ as being 'entire'. If there does not exists a $z \in \mathbb{C}$ such that $f^{\prime}(z)$ exists, we say that $f$ is 'nowhere differentiable'; such examples include:

$$
\begin{equation*}
g_{1}(z)=\operatorname{Re}(z) \quad g_{2}(z)=\operatorname{Im}(z) \quad g_{3}(z)=|z| \quad g_{4}(z)=\bar{z} \tag{1.36}
\end{equation*}
$$

Verification of these being nowhere differentiable can be shown by letting $\Delta z$ approach to 0 from the real and imaginary directions and showing that the limits are not equal; see [7] for details.

A property that we will take full advantage of is the following.
Theorem 1.1.1. (Cauchy-Riemann) If $f=u+i v$ is analytic at a point $z_{0}$, then the following system holds:

$$
\begin{equation*}
u_{x}=v_{y}, \quad v_{x}=-u_{y} \tag{1.37}
\end{equation*}
$$

Proof. Let $f=u+i v$ be analytic at $z_{0}=x_{0}+i y_{0}$. Taking the derivative in the $x$-direction, we obtain:

$$
\begin{equation*}
f^{\prime}\left(z_{0}\right)=\frac{\partial u}{\partial x}\left(x_{0}, y_{0}\right)+i \frac{\partial v}{\partial x}\left(x_{0}, y_{0}\right) \tag{1.38}
\end{equation*}
$$

Taking the derivative in the $y$-direction, we obtain:

$$
\begin{equation*}
f^{\prime}\left(z_{0}\right)=-i \frac{\partial u}{\partial y}\left(x_{0}, y_{0}\right)+\frac{\partial v}{\partial y}\left(x_{0}, y_{0}\right) . \tag{1.39}
\end{equation*}
$$

Equating real and imaginary parts of the preceding two equations gives the desired result.

A necessary and sufficient condition for $f$ to be differentiable at a point $z_{0}$ is that the CauchyRiemann equations hold, and each of the partial derivatives are continuous at $z_{0}$; one can see [21] for other properties. A useful property of differentiable complex functions is the following.

Theorem 1.1.2. A function $f$ of a complex variable $z$ is differentiable with respect to $z$ if and only if $\frac{\partial f}{\partial \bar{z}}=0$.

Proof. By the chain rule, we have:

$$
\frac{d f}{d z}=\frac{\partial f}{\partial z}+\frac{\partial f}{\partial \bar{z}} \cdot \frac{d \bar{z}}{d z}
$$

Since we know that $g(z)=\bar{z}$ is a nowhere differentiable function, the derivative $\frac{d f}{d z}$ only exists if the partial derivative of $f$ with respect to $\bar{z}$ is identically zero.

Theorem 1.1.3. If the total derivative, $\frac{d f}{d z}$, of a complex-variable function $f$ exists, it is equal to the partial derivative of $f$ with respect to $z$.

Proof. Since $\frac{d f}{d z}$ exists, we know that $\frac{\partial f}{\partial \bar{z}}=0$. Therefore from the chain-rule, we have that $\frac{d f}{d z}=\frac{\partial f}{\partial z}$.

A representation that we will take advantage of is known as the Wirtinger (representation of) derivatives, which writes the derivative with respect to $z$ in terms of the real and imaginary parts of $z$ :

$$
\begin{align*}
\frac{\partial}{\partial z} & =\frac{1}{2} \cdot\left(\frac{\partial}{\partial x}-i \frac{\partial}{\partial y}\right)  \tag{1.40}\\
\frac{\partial}{\partial \bar{z}} & =\frac{1}{2} \cdot\left(\frac{\partial}{\partial x}+i \frac{\partial}{\partial y}\right)
\end{align*}
$$

One can easily verify that these representations (Eq 1.40) satisfy (or fail) simultaneously with the validity of the Cauchy-Riemann equations. We will also refer to a function as being 'harmonic' if the real and imaginary parts of $f$ satisfy Laplace's equation:

$$
\begin{equation*}
\nabla^{2} f \equiv \frac{\partial^{2} f}{\partial x^{2}}+\frac{\partial^{2} f}{\partial y^{2}}=0 \tag{1.41}
\end{equation*}
$$

One can show that $f$ is analytic at a point $z_{0}$ if and only if $f$ has a Taylor series about $z_{0}$ given by:

$$
\begin{equation*}
f(z)=\sum_{k=0}^{\infty} \frac{f^{(k)}\left(z_{0}\right)}{k!}\left(z-z_{0}\right)^{k} \tag{1.42}
\end{equation*}
$$

If we choose $z_{0}=0$, we obtain the MacLaurin series for $f$ :

$$
\begin{equation*}
f(z)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^{k} \tag{1.43}
\end{equation*}
$$

For approximation purposes, we may only want to take the first $N+1$ terms of the Taylor
series. We will call this the $N^{t h}$-order Taylor polynomial of $f$ and is given by:

$$
\begin{equation*}
f(z) \approx f\left(z_{0}\right)+f^{\prime}\left(z_{0}\right)\left(z-z_{0}\right)+\frac{1}{2!} f^{\prime \prime}\left(z_{0}\right)\left(z-z_{0}\right)^{2}+\ldots+\frac{1}{N!} f^{(N)}\left(z_{0}\right)\left(z-z_{0}\right)^{N} \tag{1.44}
\end{equation*}
$$

For numerical purposes, we tend to want the Taylor series about the point $z$, which gives us:

$$
\begin{equation*}
f(z+\Delta z)=f(z)+\Delta z \cdot f^{\prime}(z)+\frac{(\Delta z)^{2}}{2!} f^{\prime \prime}(z)+\ldots \tag{1.45}
\end{equation*}
$$

We can also generalize this one-dimensional $(z)$ Taylor series to two complex dimensions $z_{1}, z_{2}$. Let us assume that it will be centered about $\left(z_{1}, z_{2}\right)=(0,0)$; then:

$$
\begin{align*}
f\left(z_{1}+\Delta z_{1}, z_{2}+\Delta z_{2}\right)= & f\left(z_{1}, z_{2}\right) \\
& +\frac{1}{1!} \cdot\left(z_{1} \cdot f_{z_{1}}\left(z_{1}, z_{2}\right)+z_{2} \cdot f_{z_{2}}\left(z_{1}, z_{2}\right)\right) \\
& +\frac{1}{2!} \cdot\left(\binom{2}{0} z_{1}^{2} \cdot f_{z_{1} z_{1}}\left(z_{1}, z_{2}\right)+\binom{2}{1} z_{1} z_{2} \cdot f_{z_{1} z_{2}}\left(z_{1}, z_{2}\right)+\binom{2}{2} z_{2}^{2} \cdot f_{z_{2} z_{2}}\left(z_{1}, z_{2}\right)\right)+\ldots \tag{1.46}
\end{align*}
$$

For sequences of constants $a_{k}$ and $b_{j}$, one defines the Laurent series of a function $f$ around $z=z_{0}$ by:

$$
\begin{equation*}
f(z)=\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}+\sum_{j=1}^{\infty} b_{j}\left(z-z_{0}\right)^{-j} . \tag{1.47}
\end{equation*}
$$

Here we see that the first summation is precisely the Taylor series, which is sometimes referred to as the analytic portion of the Laurent series, and the second summation gives powers on the singular portion of $z$; the second summation is also sometimes referred to as the principal part of the Laurent series. A more extensive discussion of convergence of Taylor series and a most important Laurent series can be found in [7].

We define the coefficient, $b_{1}$ of the Laurent series to be the residue of $f$ at the point $z=z_{0}$
and denote it by:

$$
\begin{equation*}
b_{1}:=\operatorname{Res}\left(f ; z_{0}\right) \tag{1.48}
\end{equation*}
$$

One can see that if the principal part of the Laurent series only goes up to $j=1$, which we then say that $z=z_{0}$ is a simple pole of $f$, then one can readily find that:

$$
\begin{equation*}
\operatorname{Res}\left(f ; z_{0}\right)=\lim _{z \rightarrow z_{0}}\left(z-z_{0}\right) f(z) \tag{1.49}
\end{equation*}
$$

If the pole has higher order, say $n$, then one can find, for $m \geq n$, that:

$$
\begin{equation*}
\operatorname{Res}\left(f ; z_{0}\right)=\frac{1}{(m-1)!} \cdot \lim _{z \rightarrow z_{0}} \frac{d^{m-1}}{d z^{m-1}}\left[\left(z-z_{0}\right)^{m} \cdot f(z)\right] \tag{1.50}
\end{equation*}
$$

One can find a summation representation of (Eq 1.50) by applying the Leibniz rule for derivatives if desired. One important application of residues is the calculation of contour integrals. Suppose that $\gamma$ is a positively oriented simple closed contour. Then if $a_{k}$ are the residues of $f$ that are located within $\gamma$, then the Cauchy-Residue theorem (see [7] for the proof) is given by:

$$
\begin{equation*}
\oint_{\gamma} f(z) d z=2 \pi i \sum_{\forall k} \operatorname{Res}\left(f ; a_{k}\right) . \tag{1.51}
\end{equation*}
$$

As an example, consider:

$$
\begin{equation*}
I=\int_{0}^{\infty} \frac{\cos x}{x^{2}+1} d x \tag{1.52}
\end{equation*}
$$

First the integral $I$, note that this integral can be rewritten as $\int_{0}^{\infty} f=\frac{1}{2} \int_{-\infty}^{\infty} f$ due to the parity of $f$. We then would like to find a contour integral in the complex plane that is related to that of $I$. We know that $\cos x$ is the real part of $e^{i x}$, hence we can consider the following
integral:

$$
\begin{equation*}
\oint_{\gamma} \frac{e^{i z}}{z^{2}+1} d z \tag{1.53}
\end{equation*}
$$

For our contour $\gamma$, we can consider the line segment from $x=-R$ to $x=R$ (which will constitute our integral as $R \rightarrow \infty$ ) and denote as $\gamma_{1}$, and the upper half of the circle of radius $R$ centered at the origin and denote as $\gamma_{2}$. Here, we orient our contour counter-clockwise. Hence:

$$
\begin{equation*}
\oint_{\gamma} \frac{e^{i z}}{z^{2}+1} d z=\int_{\gamma_{1}} \frac{e^{i x}}{x^{2}+1} d x+\int_{\gamma_{2}} \frac{e^{i z}}{z^{2}+1} d z \tag{1.54}
\end{equation*}
$$

First note that the integrand $f$ has two singularities, $z= \pm i$, where only $z=i$ lies inside $\gamma$. One can find that $\operatorname{Res}(f ; i)=\frac{e^{-1}}{2 i}$, hence $\oint_{\gamma} f=2 \pi i \cdot \frac{e^{-1}}{2 i}=\frac{\pi}{e}$. For the semicircle, we can parametrize the path via $z=R e^{i \theta}$, where $0 \leq \theta \leq \pi$. Through substitution and simplification, we can then write (Eq 1.54) as:

$$
\begin{equation*}
\frac{\pi}{e}=\int_{-R}^{R} \frac{e^{i x}}{x^{2}+1} d x+\int_{0}^{\pi} \frac{R i e^{i \theta} e^{i R \cos \theta} e^{-R \sin \theta}}{R^{2} e^{2 i \theta}+1} d \theta \tag{1.55}
\end{equation*}
$$

Note that $0 \leq \theta \leq \pi$, hence $0 \leq \sin \theta \leq 1$, hence as $R \rightarrow \infty$, one can find that the integral over $\gamma_{2}$ vanishes to 0 . After applying Euler's formula, we have:

$$
\begin{equation*}
\frac{\pi}{e}=\lim _{R \rightarrow \infty}\left(\int_{-R}^{R} \frac{\cos x}{x^{2}+1} d x+i \int_{-R}^{R} \frac{\sin x}{x^{2}+1} d x\right) \tag{1.56}
\end{equation*}
$$

Equating real parts gives us the result, $I=\frac{\pi}{2 e}$.

### 1.1.4 The Gamma Function and Integral Transforms

Consider a function $f$ on a domain $\Omega$ and let $\Omega$ be partitioned into sub-domains $\Omega_{k}$ and let $t_{k} \in \Omega_{k}$ be tags in each sub-domain. Then we define the Riemann sum $S_{n}$ by:

$$
\begin{equation*}
S_{n}=\sum_{k=1}^{n} f\left(t_{k}\right) \cdot\left(z_{k}-z_{k-1}\right) \tag{1.57}
\end{equation*}
$$

If we take the limit as $n \rightarrow \infty$, and the corresponding limit exists and is equal to $L$, we say that $f$ is 'integrable' on $\Omega$ and we call the limit the integral of $f$ on $\Omega$, and denote this by:

$$
\begin{equation*}
\lim _{n \rightarrow \infty} S_{n}=L:=\int_{\Omega} f(z) d z \tag{1.58}
\end{equation*}
$$

One of the complex variable functions that is defined by an integral is the gamma function [28], which is given by:

$$
\begin{equation*}
\Gamma(z)=\int_{0}^{\infty} t^{z-1} e^{-t} d t \tag{1.59}
\end{equation*}
$$

This is also known as the Mellin transform of $e^{-t}$, whose properties and applications can be found in [18]. One of the simplest properties of the gamma function is its connection to the factorial:

$$
\begin{equation*}
\Gamma(x)=(x-1)!, x \in \mathbb{N} \tag{1.60}
\end{equation*}
$$

Substituting $x+i y$ for $z$ in the definition of $\Gamma(z)$ gives us:

$$
\begin{equation*}
\Gamma(z)=\left[\int_{0}^{\infty} \cos (y \ln t) t^{x-1} e^{-t} d t\right]+i\left[\int_{0}^{\infty} \sin (y \ln t) t^{x-1} e^{-t} d t\right] \tag{1.61}
\end{equation*}
$$

If one were to plot the real part on the $z$-axis and the imaginary part via a color contour, one will obtain a relatively flat plane since the growth of the factorial is extreme compared to preceding numerical values, but the color bar states that the argument does change
depending on the value in question. An alternative to graphing complex functions is to graph the modulus of the function on the z-axis and graph the argument of the function via a color contour. There are several other integral-defined functions such as the sine


Figure 1.5: Modulus-Argument Graph of the Gamma Function
integral, error function, Bessel function, and Fresnel integral, to name a few, but we will only introduce ones and their properties when they arise in our results and analysis. Other integral-represented functions that have a strong purpose are ones that transform functions of one variable into a function of another variable; in particular, the Fourier transform maps a function of time $t$ into a function of frequency $\omega$. One transformation is known as the Laplace transform, which is defined by:

$$
\begin{equation*}
\mathcal{L}_{f}(\xi)=\int_{0}^{\infty} f(x) e^{-x \xi} d x \tag{1.62}
\end{equation*}
$$

which exists provided that $f$ is of exponential order (see [22] for details). The Laplace transform is used in several different applications, one of them stemming from their application to solve differential equations, say $y^{\prime}(x)=f(x, y)$ by taking advantage of the property:

$$
\begin{equation*}
\mathcal{L}\left\{y^{\prime}\right\}=\xi \mathcal{L}\{y\}-y(0) \tag{1.63}
\end{equation*}
$$

One can recursively apply this procedure to solve any order ordinary differential equation initial value problem using this strategy, to obtain an equation of the form:

$$
\mathcal{L}\{y\}=F(\xi) \quad \rightarrow \quad y=\mathcal{L}^{-1}\{F\}
$$

Unless the inverse Laplace transform of $F$ is trivial, it is sometimes difficult to find the exact elementary representation of $y$ (if it even exists). We will investigate and apply the 2-dimensional analogue of the Laplace transform and denote it accordingly by:

$$
\begin{equation*}
\mathcal{L}_{f(x, y)}\left(\xi_{1}, \xi_{2}\right)=\int_{0}^{\infty} \int_{0}^{\infty} f(x, y) e^{-x \xi_{1}-y \xi_{2}} d x d y \tag{1.64}
\end{equation*}
$$

We will discuss properties and some identities when the appropriate time arises. Another important transformations is the Fourier transform:

$$
\begin{equation*}
\mathcal{F}_{f}(\xi)=\int_{-\infty}^{\infty} f(x) \cdot e^{-2 \pi i x \xi} d x=F(\xi) \tag{1.65}
\end{equation*}
$$

To go in the reverse direction, the inverse Fourier transform is defined similarly (ignoring the normalization factor):

$$
\begin{equation*}
\mathcal{F}_{F}^{-1}(x)=\int_{-\infty}^{\infty} F(\xi) e^{2 \pi i x \xi} d \xi=f(x) \tag{1.66}
\end{equation*}
$$

Since the Fourier transform maps a real-variable function into the complex domain (traditionally from the time to frequency domains), it has a real part and imaginary part. The modulus of the Fourier transform is traditionally referred to as the 'Fourier spectrum' of $f$, and the argument of the Fourier transform is referred to as the 'phase angle' of $f$, each of which are defined in the intuitive fashion. More details on the Laplace transform, its existence and special properties can be found in [22], and another transformation known as the Mellin transform is described well in [18], which we will leave to the interested reader to
explore [5].

The Fourier transform satisfies several properties that makes it useful, the first of them being a linear operator. The second property is the shift theorem, which states that a horizontal shift in the domain results in a amplitude change of the Fourier transform:

$$
\begin{equation*}
\mathcal{F}\left[f\left(x-x_{0}\right)\right](\xi)=e^{-2 \pi i \xi x_{0}} \cdot F(\xi) \tag{1.67}
\end{equation*}
$$

If $f$ is two-dimensional and $f$ is rotated by some angle $\phi$, then the Fourier transform will also be rotated by $\phi$. Since the Fourier transform is defined to be a spatial transformation, it has no effect on time-dependant partial derivatives:

$$
\begin{equation*}
\mathcal{F}\left[f_{t}\right](\xi)=\frac{\partial}{\partial t} \mathcal{F}[f](\xi) \tag{1.68}
\end{equation*}
$$

Using integration by parts inductively on the spatial partial derivatives of a function, one obtains the ever-so-useful identity:

$$
\begin{equation*}
\mathcal{F}\left[\frac{\partial^{n} f}{\partial x^{n}}\right](\xi)=(i \xi)^{n} \cdot \mathcal{F}[f](\xi) \tag{1.69}
\end{equation*}
$$

which intuitively transforms differential representations into algebraic ones.

### 1.1.5 Recent Results in Complex Analysis

Aside from the things mentioned above, another field that is highly important is the theory of contour integration. Many texts discuss this field including [7], [8], and [11]. The theoretical aspects go as far as the theory of complex manifolds [13] to theory of functions of several complex variables [14]. We will investigate the several-variable portion of complex analysis, but only from the numerical perspective.

### 1.2 Numerical Analysis of Functions

Since the calculus required to solve some problems may be intensive, it is highly useful to analyze the numerical methods to solve mathematical problems. Take for example (Eq 1.61); these improper integrals can not be evaluated analytically, requiring numerical approximations of the integrals (say, by Riemann sums).

### 1.2.1 Algebraic Equations

Due to the highly complex nature of the world we live in (note the multiple interpretations of the word 'complex' in this sentence), there are many equations in which the exact solution is not easily or even possibly obtainable. Take for example the equation:

$$
3^{-x}=4 x
$$

We know that this equation has exactly one solution since the graphs of $y=3^{-x}$ and $y=4 x$ intersect exactly once, but the solution of this equation does not exist in terms of elementary functions. To solve, consider an equation of the form $\varphi(x)=0$, and let $\varphi$ have properties that allow $\varphi$ to have a Taylor series. Then the first-order Taylor polynomial approximation is given by:

$$
\begin{equation*}
\varphi(x) \approx \varphi\left(x_{0}\right)+\left(x-x_{0}\right) \varphi^{\prime}\left(x_{0}\right) \tag{1.70}
\end{equation*}
$$

Since we desire $\varphi(x) \equiv 0$, solving for $x$ gives us:

$$
\begin{equation*}
x \approx x_{0}-\frac{\varphi\left(x_{0}\right)}{\varphi^{\prime}\left(x_{0}\right)}:=x_{1} \tag{1.71}
\end{equation*}
$$

This then motivates an iterative process; for some initial guess $x_{0}$, for $k>0$ we have:

$$
\begin{equation*}
x_{k+1} \approx x_{k}-\frac{\varphi\left(x_{k}\right)}{\varphi^{\prime}\left(x_{k}\right)} \tag{1.72}
\end{equation*}
$$

which we will refer to as the Newton-Raphson (N. R.) method. Since $\varphi$ is known throughout its domain, we can approximate its derivative with a second-order central-difference approximation (or higher if desired):

$$
\begin{equation*}
\varphi^{\prime}\left(x_{k}\right) \approx \frac{\varphi\left(x_{k}+h\right)-\varphi\left(x_{k}-h\right)}{2 h} \tag{1.73}
\end{equation*}
$$

where $h$ is sufficiently small. It is clear to see that if $x_{0} \in \mathbb{R}$, then the approximated solution $x_{k}$ will also be real. On the other hand, if the initial guess is complex, then the N.R. method should converge to a complex value. As an example, consider the following second-degree polynomial equation:

$$
\begin{equation*}
5 z^{3}-(18-33 i) z^{2}-(17+60 i) z+(6+43 i)=0 \tag{1.74}
\end{equation*}
$$

By the Fundamental Theorem of Algebra, there exists 3 complex solutions to this equation. Since all of the coefficients of the polynomial are algebraic, it is safe to choose a transcendental intial guess, say $x_{0}=\pi \cdot i$. This assumption is easily verifiable since the coefficients of $f$ are algebraic, the roots of $f^{\prime}$ (which is where N.R. fails) are also algebraic. Applying NewtonRaphson, we obtain the first solution to be $z_{1}=\frac{3}{5}+\frac{2}{5} i$. Since this is a polynomial, we can use synthetic division of $z_{1}$ into $\varphi(z)$ to obtain our remainder polynomial. Applying Horner's method (sometimes referred to simply as synthetic division), we obtain Figure 1.6. Applying

$$
\begin{aligned}
& \left(\frac{3}{5}+\frac{2}{5} i\right): \quad 5-18+33 i \quad-17-60 i \quad 6+43 i \\
& 0 \quad 3+2 i \quad-23+15 i \quad-6-43 i \\
& \text { (5) }(-15+35 i)(-40-45 i) 0
\end{aligned}
$$

Figure 1.6: Horner's Method - Step 1
N.R. on the second-degree quotient, we obtain the second solution to be $z_{2}=1-2 i$. Applying

Horner's method again, we obtain Figure 1.7. Applying N.R. on the linear quotient (which

$$
\begin{array}{cccc}
(1-2 i) & : & 5 & -15+35 i
\end{array}-40-45 i ~\left(\begin{array}{ccc} 
& 0 & 5-10 i
\end{array} 40+45 i\right.
$$

Figure 1.7: Horner's Method - Step 2
can be by-passed to save computation time if necessary since the closed form is unique), we obtain the third solution to be $z_{3}=(2-5 i)$. Let $p_{c}=[5,-18+33 i,-17-60 i, 6+43 i]$ and $q_{c}$ represent the vectors of the coefficients of the original polynomial $p$ and the obtained polynomial $q(z)=\prod_{k=1}^{n-1}\left(x-z_{k}\right)$. We can measure the accuracy of a factorization via the following norm:

$$
\begin{equation*}
\|p-q\|:=\sqrt{\sum_{k=1}^{n-1}\left(p_{c_{k}}-q_{c_{k}}\right)^{2}} \tag{1.75}
\end{equation*}
$$

Our example gives us the exact factorization, hence the norm is trivially 0 in our case.
Consider a function $f(x)$ and a point $p \in \operatorname{dom}(f)$. Then we say that $p$ is a fixed point for $f$ if $f(p)=p$. We mention this because it gives rise to a method of being able to solve equations for a variable $x$ that can be written in the form $x=f(x)$, where $f$ is an algebraic function. For the function at the beginning of the section we know that for the equation:

$$
\begin{equation*}
x=\frac{1}{4} \cdot 3^{-x}:=f(x), \tag{1.76}
\end{equation*}
$$

we know that whatever solution $x$ makes this true, is a fixed point for the function $f$. One can show, using the mean value theorem, that if $f$ exists on some interval $(a, b)$ and the derivative $f^{\prime}$ exists on this interval and satisfies the property that $\left|f^{\prime}(x)\right| \leq k<1, \forall x \in(a, b)$ (this is sometimes referred to as Lipshitz continuous), then the fixed point $p$ exists in $(a, b)$ and it
is also unique.

We know that our function $f$ has the property that $\left|f^{\prime}(0)\right|=\frac{\ln (3)}{4}$ and the function $\left|f^{\prime}\right|$ is always decreasing for $(0, \infty)$, therefore there is a unique fixed point for $f$, and hence $x=f(x)$ has a unique solution $p \in(0, \infty)$. Once you find the interval $(a, b)$ for which the fixed point $x=p$ exists, we must then find $p$. Suppose we give an initial guess $x_{0} \in(a, b)$ and consider the sequence defined by:

$$
x_{k+1}=f\left(x_{k}\right), \quad x_{0} \text { is initial guess },
$$

and define the error term $e_{k+1}$ to the exact solution $p$ by:

$$
\begin{aligned}
e_{k+1} & :=\left|x_{k+1}-p\right| \\
& \equiv\left|f\left(x_{k}\right)-f(p)\right| .
\end{aligned}
$$

Using the mean value theorem, we have:

$$
\begin{aligned}
e_{k+1} & =\left|f^{\prime}(p)\right| \cdot\left|x_{k}-p\right| \\
& =k \cdot e_{k} .
\end{aligned}
$$

If the point $p$ is unique, then $k<1$, hence $e_{k+1}$ is a decreasing Cauchy sequence in which converges to $p$. This method of solving equations is known as the fixed-point iteration method.

### 1.2.2 Ordinary Differential Equations

The field of ordinary differential equations (ODE's) have several methods that solve initial value problems. To name a few:

- Euler, Mid-Point, Taylor
- Backward Euler, Adams-Moulton, Backward Differentiation Formulas (BDF).

The first set is referred to as 'explicit' (being able to calculate the unknown directly), and the second set 'implicit' (having to solve an equation for the unknown). Each method has their set of advantages, disadvatanges, and convergence criterion which is developed beautifully in [10] and [19]. A class of methods that tend to be occasionally preferred are 'adaptive methods', in which the interval size $\Delta x$ changes depending on the behavior of the solution; that is, if the solution changes rapidly, go back and make your $\Delta x$ smaller (and vice versa). We will avoid the use of such methods due to the computational cost they have for arbitrarily large $N$. For the solutions of initial value problems

$$
\begin{equation*}
\frac{d y}{d x}=f(x, y), \quad y\left(x_{0}\right)=\alpha \tag{1.77}
\end{equation*}
$$

we will use the standard Runge-Kutta of Order 4 method (RK4) due to its simplicity in implementation and high order of accuracy; the algorithm is given below:

$$
\begin{align*}
w_{1} & =\alpha \\
k_{1} & =h \cdot f\left(x_{k}, w_{k}\right) \\
k_{2} & =h \cdot f\left(x_{k}+\frac{h}{2}, w_{k}+\frac{1}{2} k_{1}\right) \\
k_{3} & =h \cdot f\left(x_{k}+\frac{h}{2}, w_{k}+\frac{1}{2} k_{2}\right)  \tag{1.78}\\
k_{4} & =h \cdot f\left(x_{k+1}, w_{k}+k_{3}\right) \\
w_{k+1} & =w_{k}+\frac{1}{6}\left(k_{1}+2 k_{2}+2 k_{3}+k_{4}\right), k=1,2, \ldots, N+1
\end{align*}
$$

If such a case may arise where the RK4 method yields high error (which occurs in the study of stiff equations), we may switch to adaptive methods or BDF's temporarily [27], but will mention so if that switch is made. For the first-order system of initial value problems, one can vectorize the Runge-Kutta method. and for the $n^{\text {th }}$ order ODE, one can use the substitutions $u_{1}=y, \ldots, u_{k}=y^{(k-1)}$, differentiate each $u$ and re-substitute to obtain a system of first-order

IVP's of $u$, which can then be solved for $y$.

### 1.2.3 Partial Differential Equations

For partial differential equations, we will focus on (at least) two approaches. The first will be finite-difference methods, where the partial derivatives are approximated using finite differences. We will represent the solution value $u\left(x_{n}, t_{k}\right)$ by $u_{n}^{k}$. To analyze the stability of finite-difference methods we will use Von-Neumann analysis [17], which begins by replacing $u_{n}^{k}$ by $\lambda^{k} e^{i n \Delta x \Delta t}$, where $\Delta x$ and $\Delta t$ are the spatial- and time-step length and $\lambda$ the eigenvalue of the finite-difference method. Once replacement is made, we obtain an expression of $\lambda$ in terms of $\Delta t$ and $\Delta x$ (and other constants of the PDE ) and determine values for which $|\lambda| \leq 1$, which produces stability. Typically the forward-difference in time is unstable, backward-difference is unconditionally stable, and central-difference is conditionally stable. Since we will be considering complex-variable PDE's, we will prove stability to verify whether these conditions carry over to the complex case. As an example of solving PDE's, consider the wave equation in one spatial dimension:

$$
\begin{equation*}
u_{t}=\kappa u_{x x} . \tag{1.79}
\end{equation*}
$$

Approximating $u_{t}$ with forward-difference (for simplicity) and $u_{x x}$ with central, we obtain:

$$
\begin{equation*}
u_{n}^{k+1}=u_{n}+\kappa \frac{\Delta t}{(\Delta x)^{2}}\left(u_{n+1}^{k}-2 u_{n}^{k}+u_{n-1}^{k}\right) . \tag{1.80}
\end{equation*}
$$

Applying Von-Neumann analysis and simplifying gives us the equation:

$$
\begin{equation*}
\lambda=1+\kappa \frac{\Delta t}{(\Delta x)^{2}}[2 \cos (\Delta t \Delta x)-2] . \tag{1.81}
\end{equation*}
$$

If we define $r:=2 \kappa \frac{\Delta t}{(\Delta x)^{2}}$ and assume that $\kappa>0$, then $r>0$ as well. For stability $\lambda \leq 1$, hence $r[\cos (\Delta x \Delta x)-1]<0$. Since $\cos (\Delta x \Delta t) \leq 1$ for any $\Delta x, \Delta t$, we have that this finite-
difference scheme will be stable for any parameter values.

The second method that we will generalize to complex PDE's is spectral methods. Since we know that the derivative operator is linear, we can represent the heat equation in operator form:

$$
\begin{equation*}
u_{t}=\kappa D^{2}(u), \tag{1.82}
\end{equation*}
$$

where $D^{2}=D(D)=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)$. Discretizing $u$ into a vector, it remains to obtain a matrix representation of $D$. One discretization of $D$ is known as the Chebyshev-spectral differentation matrix, which has the structure shown in Figure 1.7. The motivation and derivation of this

$D_{N}=$| $\frac{2 N^{2}+1}{6}$ | $2 \frac{(-1)^{j}}{1-x_{j}}$ | $\frac{1}{2}(-1)^{N}$ |
| :--- | :--- | :--- |
| $-\frac{1}{2} \frac{(-1)^{i}}{1-x_{i}}$ | $\frac{(-1)^{i+j}}{x_{i}-x_{j}}$ |  |
| $-\frac{-x_{j}}{2\left(1-x_{j}^{2}\right)}$ | $\frac{1}{2} \frac{(-1)^{N+i}}{1+x_{i}}$ |  |
| $-\frac{1}{2}(-1)^{N}$ | $-2 \frac{(-1)^{N+j}}{1+x_{j}}$ | $-\frac{2 N^{2}+1}{6}$ |

Figure 1.8: Spectral Differentiation Matrix
matrix can be found in [26]. Applying a forward difference approximation to $u_{t}$, we obtain:

$$
\begin{equation*}
u_{n}^{k+1}=u_{n}^{k}+\kappa \Delta t \cdot D^{2}(u) \tag{1.83}
\end{equation*}
$$

This approach is very easy to implement compared to finite-difference methods and gives beautiful results as well. Stability for the spectral method can easily be determined by ensuring that the eigenvalues lie within the unit circle [15]. The difficulty of spectral methods
for extension to our case will be the derivation of the spectral representation of the complex derivative (as we will discuss later).

### 1.2.4 Recent Results in Numerical Analysis

A concise and well-organized summary of numerical methods, both new and old, can be found in [1]. This source covers the discussion of stiff- and mild stiff- ODE's, single step, multi-step, implicit, explicit, pseudo Runge-Kutta, Fehlberg, and implicit Runge-Kutta methods; discussion of their stability is also included. The use of spectral methods has been famous for decades [4]. The highlight of spectral methods is primarily in the solution of non-linear partial differential equations, such as the Navier-Stokes (N.S.) equation [2] and the Fokker-Planck-Landau (FPL) equation [16], which is used in plasma physics. Due to its complexity, the field of complex partial differential equations is a thin field, but some theoretical analysis has been established [9] [3]. Some partial differential equations have complex parameters, such as the non-linear Schrodinger-equation [6]; this equation has seen vast applications in optics and quantum physics. Since the analytical and theoretical nature of complex PDE's has proven to be vastly complex, we investigate the numerical nature of those equations and their solutions.

## Chapter 2 <br> COMPLEX-VARIABLE FIRST-ORDER DIFFERENTIAL EQUATIONS

In this chapter, we begin the study of complex-variable differential equations. We will begin with the standard sequence of differential equations of two different types: $\omega^{\prime}=f(z)$ and $\omega^{\prime}=f(z, \omega)$. Our methodology will be based on the theoretical method presented. The results will be based and discussed on accuracy to exact solution (when available). The results in (2.1) and (2.2) will be based on the geometric nature of the surface for qualitative analysis, and also give error-based measures for a quantitative analysis. The results in (2.3) will discuss the problems in (2.1) and (2.2) in a qualitative approach. Section (2.4) will follow up to (2.1) in regards to the theoretical aspects.

### 2.1 One-Dimensional Complex Spatial IVP's, where $f=f(z)$

The first problem that we will be interested in is the following initial value problem (IVP):

$$
\begin{equation*}
\frac{d \omega}{d z}=f(z), \omega\left(z_{0}\right)=\omega_{0} \tag{2.1}
\end{equation*}
$$

Here, $z=x+i y$ is the standard complex number with $x, y \in \mathbb{R}$ and $\omega=u+i v$ with $u, v$ being real valued functions of the variables $x, y$. Since we are considering the initial value problem of $\omega$ with respect to $z$, it is necessary for $\omega$ to be analytic in the domain of definition. Hence for the solution of the first-order initial value problem to exist, it follows as an immediate result that $u, v$ satisfy the Cauchy-Riemann equations. By properties of the differential operator (discussed in section 1.1.3), we know that the existence of solution to our initial value problem exists and is explicitly given by:

$$
\begin{equation*}
\omega(z)=\omega_{0}+\int_{z_{0}}^{z} f(t) d t \tag{2.2}
\end{equation*}
$$

therefore the problem is well-posed. This does not imply that the solution is unique, as we will discuss at the end of this chapter. From Theorem 1.1.3, the initial value problem is equivalent to the following complex partial differential equation:

$$
\begin{equation*}
\frac{\partial \omega}{\partial z}=f(z) \tag{2.3}
\end{equation*}
$$

For any function $f=f(z)$, we can always obtain the real and imaginary parts of $f$ since it is a function of $z$ only. For the time being, we define $\operatorname{Re}(f)=a(x, y)$ and $\operatorname{Im}(f)=b(x, y)$. If we decompose our unknown function $\omega$ into its corresponding unknown real and imaginary parts $u, v$, we obtain the following partial differential equation:

$$
\begin{equation*}
\left(\frac{\partial u}{\partial z}+i \frac{\partial v}{\partial z}\right)=a+b i . \tag{2.4}
\end{equation*}
$$

From (Eq. 1.23), we can rewrite the complex differential operator $\frac{\partial \omega}{\partial z}$ in terms of $u, v, x, y$ to give us the following system:

$$
\begin{align*}
& u_{x}+v_{y}=2 a  \tag{2.5}\\
& v_{x}-u_{y}=2 b
\end{align*}
$$

Since $u, v$ satisfy the C.R. equations, in equation (2.5) replace $v_{y}$ with $u_{x}$ and $v_{x}$ with $-u_{y}$ to obtain a partial-system for $u$ :

$$
\begin{equation*}
u_{x}=a(x, y) \quad u_{y}=-b(x, y) \tag{2.6}
\end{equation*}
$$

Similarly, in equation (2.5) replace $u_{x}$ with $v_{y}$ and $u_{y}$ with $-v_{x}$ to obtain a partial-system for $v$ :

$$
\begin{equation*}
v_{x}=b(x, y) \quad v_{y}=a(x, y) \tag{2.7}
\end{equation*}
$$

As a demonstration, so solve for the real-part of our solution, $u$, we can use the system given in (Eq 2.6) to solve in each direction. To solve numerically, we apply a first-order finite difference to (Eq 2.6) to obtain:

$$
\begin{align*}
& u(x+h, y)=u(x, y)+h \cdot a(x, y)  \tag{2.8}\\
& u(x, y+h)=u(x, y)-h \cdot b(x, y)
\end{align*}
$$

Our initial condition is $\omega\left(z_{0}\right)=\omega_{0}$, where $z_{0}=x_{0}+i y_{0}$ and $\omega_{0}=u_{0}+i v_{0}$. Since we are given these initial conditions, one can obtain the solution of $u$ along the solution plane $y=y_{0}$ by:

$$
\begin{align*}
u\left(x_{0}+h, y_{0}\right) & =u\left(x_{0}, y_{0}\right)+h \cdot a\left(x_{0}, y_{0}\right) \\
u\left(x_{0}+2 h, y_{0}\right) & =u\left(x_{0}+h, y_{0}\right)+h \cdot a\left(x_{0}+h, y_{0}\right)  \tag{2.9}\\
& \vdots \\
u\left(x_{f}, y_{0}\right) & =u\left(x_{f}-h, y_{0}\right)+h \cdot a\left(x_{f}-h, y_{0}\right) .
\end{align*}
$$

Here, $x_{f}$ represents a user-defined end of the computational domain of $x$. We can then obtain the solution along $y=y_{0}+h:=y_{1}$ from (Eq 2.8.2) to be:

$$
\begin{equation*}
u\left(x_{0}, y_{k+1}\right)=u\left(x_{0}, y_{k}\right)-h \cdot b\left(x_{0}, y_{k}\right), k=1,2, \ldots \tag{2.10}
\end{equation*}
$$

We can iterate this process for all $y$ until $y=y_{f}$, where $y_{f}$ is the user-defined end of the computational domain of $y$. Using the system in (Eq 2.7), we can use this same procedure to find the solution $v$ on $x \in\left[x_{0}, x_{f}\right], y \in\left[y_{0}, y_{f}\right]$. At this point, we will have the solution of $u, v \in \mathcal{D}(x, y)$ and hence we have the solution $\omega \in \mathcal{D}(z)$. In this process, we initialized through the $y$ direction and solved through the $x$ direction; we will refer to this as the "Left-to-Right" method (or simply LR). A diagram illustrating the LR method and a similar and equivalently computational method - initialize through x , solve through y , which I will refer to as the "Bottom-to-Top" (BT) method - is shown in Figure 2.1. The two methods are theoretically the same, but we will aim to verify that the produce the same numerical results.


Initialize y, Solve through x (LR)


Initialize $x$, Solve through y (BT)

Figure 2.1: Solution Strategies for One Complex Spatial Dimension

To demonstrate the aforementioned method(s) of solving $\omega^{\prime}=f(z)$, we will choose the following examples for which the solution is easily accessible for verification. Consider:

$$
\begin{equation*}
\frac{d \omega}{d z}=2 z, \quad \omega(2+i)=-3+7 i \tag{2.11}
\end{equation*}
$$

Since $z_{0}=2+1 i$, we will choose our computational domain to be $\mathcal{D}=[2,7] \times[1,6]$. We also will use a step size of $h=0.05$ in both the $x$ and $y$ directions (this can be changed based on user preference). Implementing the method described in this section, we obtain the following results.


Figure 2.2: Real/Imaginary Parts for Solution of $\omega^{\prime}=2 z$ via LR-Method

To determine the accuracy of these numerical results, one can find that the exact solution to Eq. (2.11) is given by

$$
\begin{align*}
\omega(z)=z^{2}-6+3 i & \rightarrow u(x, y)=\operatorname{Re}(\omega)=x^{2}-y^{2}-6  \tag{2.12}\\
& \rightarrow v(x, y)=\operatorname{Im}(\omega)=2 x y+3
\end{align*}
$$

We can graph the exact solutions on on computational domain $\mathcal{D}$ and obtain the geometry of the solution that our numerical solution should resemble. Before implementation on another


Figure 2.3: Real/Imaginary Parts for Exact Solution of $\omega^{\prime}=2 z$
example, it is to wonder whether the BT method gives the same result (it should) as the LR method. Before (Eq 2.6-7), we initialized $y$ with the equations $u_{y}=-b$ and $v_{y}=a$, for the BT method, we will now initialize $x$ with the equations $u_{x}=a$ and $v_{x}=b$.

Once initialization is complete, we can use the alternate set of equations to solve through the $y$ direction. Performing this sequence gives the following results.


Figure 2.4: Real/Imaginary Parts for Solution of $\omega^{\prime}=2 z$ via BT-Method

Based on the surfaces obtained from our numerical solution, it qualitatively seems that the LR and BT methods give us the same solutions. To determine if they are actually equal, we can consider the following difference surfaces:

$$
\begin{align*}
D_{u} & :=\left|u_{L R}-u_{B T}\right|  \tag{2.13}\\
D_{v} & :=\left|v_{L R}-v_{B T}\right|
\end{align*}
$$

If these methods give the same solution, then we know that $D_{u}=D_{v}=0, \forall(x, y)$. In Figure 2.5 , the graphs for $D_{u}$ and $D_{v}$ are presented.


Figure 2.5: Difference Surfaces for the LR and BT Solutions of $u$ and $v$ (Ex. 1)

In terms of quantitative differences, we see that the LR and BT methods for computing the solution of $\operatorname{Re}(\omega)=u$ both give the same the same solution. This can be seen by noting that the maximum of $\left|u_{L R}-u_{B T}\right| \approx 10^{-14} \equiv=0$. As for the numerical solution to $\operatorname{Im}(\omega)=v$, we see that the maximum of $\left|v_{L R}-v_{B T}\right| \approx 0.5$. It would be careless to say that this is approximately equal to 0 in the same sense as the approximation to $u$, but one can at least claim that since the actual solution of $v$ tends to about 80 whereas $u$ tends to about 40, then the differences in the approximation may tend to be a bit higher. Another interesting observation is the symmetric absolute-value plane of $D_{v}$. To better understand this, consider the three non-initial vertices of $\mathcal{D}$ and the exact and numerical values (Figure 2.6).

| $(x, y)$ | $u$ | $u_{L R}$ | $u_{B T}$ | $v$ | $v_{L R}$ | $v_{\text {BT }}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $(7,1)$ | 41.3025 | 41.550 | 41.550 | 16.9000 | 17.395 | 16.900 |
| $(2,6)$ | -37.4025 | -37.650 | -37.650 | 26.8000 | 26.800 | 27.295 |
| $(7,6)$ | 6.9000 | 6.900 | 6.900 | 85.7050 | 86.2 | 86.2 |

Figure 2.6: Error Analysis for Domain Corners (Ex. 1)

As mentioned before, the approximations for $u$ from both LR and BT give the same results; but for $v$ we see something interesting. Keep in mind that our domain of definition is $\mathcal{D}=[2,7] \times[1,6]$. The LR method begins by initializing the $y$, hence obtains an accurate value for $(2,6)$. The BT method begins by initializing the $x$, hence obtains an accurate value for $(7,1)$. As we see the methods give approximately correct answers for the opposing corners and a relatively accurate approximation to the terminal corner of $\mathcal{D}$. We will take advantage of these observations at a later time, and instead will improve the accuracy of these methods in the next section. As a secondary demonstration of this method, consider:

$$
\begin{equation*}
\frac{d \omega}{d z}=\frac{1}{z}, \quad \omega(1)=2-5 i \tag{2.14}
\end{equation*}
$$

whose solution (as shown in Eq. (1.15)) is given by:

$$
\begin{align*}
\omega(z)=\ln (z)+2-5 i & \rightarrow u(x, y)=\operatorname{Re}(\omega)=\ln \sqrt{x^{2}+y^{2}}+2  \tag{2.15}\\
& \rightarrow v(x, y)=\operatorname{Im}(\omega)=\arctan \left(\frac{y}{x}\right)-5
\end{align*}
$$

We will consider the computational domain $\mathcal{D}=[1,4] \times[0,3]$.


Figure 2.7: Real/Imaginary Parts for Exact Solution of $\omega^{\prime}=\frac{1}{z}$

As was the first example, the following solutions will be calculated with the same exact method as was described with $h=0.05$.


Figure 2.8: Real/Imaginary Parts for Solution of $\omega^{\prime}=\frac{1}{z}$ via LR-Method



Figure 2.9: Real/Imaginary Parts for Solution of $\omega^{\prime}=\frac{1}{z}$ via BT-Method

As was the first example, the LR and BT methods qualitatively give the same solution surface; but as Figure 2.10 shows, the solutions are (to the fraction of a decimal) different from one another.


Figure 2.10: Difference Surfaces for the LR and BT Solutions of $u$ and $v$ (Ex. 2)

Since the difference surfaces for $u$ and $v$ are not-exactly zero (maximum of approximately 0.045 for $u$ and 0.035 for $v$ ), then it becomes of interest which method is closer to the exact solution. Since the curvature of the exact surfaces are not quite constant, we will choose both the corners of $\mathcal{D}$ and the mid-points of the boundary of $\mathcal{D}$ as well for comparison (see Figure 2.11).

| $(x, y)$ | $u$ | $u_{L R}$ | $u_{\text {BT }}$ | $v$ | $v_{L R}$ | $v_{\text {BT }}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $(1,1.5)$ | 2.5893 | 2.5776 | 2.5447 | -4.0172 | -4.0500 | -4.0716 |
| $(1,3)$ | 3.1513 | 3.1436 | 3.1000 | -3.7510 | -3.7785 | -3.7923 |
| $(2.5,0)$ | 2.163 | 2.8805 | 2.8815 | -5 | -5.0285 | -5 |
| $(2.5,1.5)$ | 3.0700 | 3.0432 | 3.0308 | -4.4596 | -4.4764 | -4.4770 |
| $(2.5,3)$ | 3.3623 | 3.3480 | 3.3225 | -4.1239 | -4.1390 | -4.1381 |
| $(4,0)$ | 3.3863 | 3.3541 | 3.3552 | -5 | -5.0362 | -5 |
| $(4,1.5)$ | 3.4521 | 3.4271 | 3.4190 | -4.6412 | -4.6604 | -4.6530 |
| $(4,3)$ | 3.6094 | 3.5952 | 3.5754 | -4.3565 | -4.3735 | -4.3668 |

Figure 2.11: Error Analysis for Domain Corners and Boundary Mid-points (Ex. 2)

As we see, the LR and BT methods take turns throughout the domain as to which is the better approximation. Again, we will take advantage of these observations at a later time.

### 2.2 Accuracy Improvements via Step Size and Finite-Difference Methods

So far in this dialogue, we have been focused on solving the equation:

$$
\frac{d \omega}{d z}=f(z), \quad \omega\left(z_{0}\right)=\omega_{0}
$$

By using the properties of analytic functions, we realized that solving this problem for $\omega=u+i v$ was equivalent to solving the system:

$$
\begin{array}{ll}
u_{x}=\operatorname{Re}(f) & u_{y}=-\operatorname{Im}(f)  \tag{2.16}\\
v_{x}=\operatorname{Im}(f) & v_{y}=\operatorname{Re}(f)
\end{array}
$$

To solve these first-order PDE's, we used a first-order difference approximation for the partial derivatives, where:

$$
\begin{equation*}
u_{x} \approx \frac{u(x+h, y)-u(x, y)}{h} \quad u_{y} \approx \frac{u(x, y+h)-u(x, y)}{h} \tag{2.17}
\end{equation*}
$$

and similar for $v_{x}$ and $v_{y}$. As we know, these approximations for the partial derivatives are first-order and can easily be derived from the Taylor series.

Regardless that we only used a first-order approximation for the derivative, we realized that the solutions that we obtained for $u$ and $v$ were almost identical to that of the exact solutions, up to a small amount of error, for both the LR and BT methods of solving through the computational domain $\mathcal{D}$. This section is devoted to the improvement of the solutions and the measurement and reduction of the error.

We will use the same two examples as were mentioned in the previous section and refer to them as "Example 1" (Eq 2.11) and "Example 2" (Eq 2.14), respectively. For the differential equations $\omega^{\prime}=f$, we will refer to the real/imaginary part of the exact solutions simply as $u$ and $v$, we will refer to the real/imaginary parts of the numerical LR solution as $u_{L R}$ and $v_{L R}$, and the real/imaginary parts of the numerical BT solution will be denoted as $u_{B T}$ and $v_{B T}$. We will measure the quantify the measure of error, $E$, using two different measures. The first measure will be defined by taking the $L_{2}$ norm of the difference of the numerical solution and the exact solution. For example:

$$
\begin{equation*}
E_{u L R}^{L 2}:=\left\|u-u_{L R}\right\|_{2} \equiv \sqrt{\sum \sum\left|u-u_{L R}\right|^{2}} \tag{2.18}
\end{equation*}
$$

The second measure will be defined by taking the relative percent difference (RPD) between the numerical solution and the exact solution. For example:

$$
\begin{equation*}
E_{u L R}^{R P D}:=2 \cdot \frac{u-u_{L R}}{|u|+\left|u_{L R}\right|} . \tag{2.19}
\end{equation*}
$$

Here, we define $E^{R P D}=0$ if the denominator of this expression is equal to 0 . Also note that the RPD measure of error takes into account whether or not the numerical solution is dominantly greater than the exact solution or dominantly less solution, where the value always lies between $-2 \leq R P D \leq 2$.

To begin, let us consider the numerical solution of both Example 1 and Example 2 with $h=0.1$. The value of $h$ may be desired to be "large" compared to standard methods due to memory constraints, but as technology improves this concern will vanish. Computing both the $L_{2}$ and RPD error measures for both examples with $h=0.1$ gives us the following results.

In terms of understanding, we desire both of these errors to be close to 0 as possible. The value of the $L_{2}$ norm is not bounded above, so the interpretation of it depends solely on the

| Example $1 \mathrm{w} / \mathrm{h}=0.1$ | LR Method | BT Method | Example 2 w/h $=0.1$ | LR Method | BT Method |
| :---: | :--- | :--- | :---: | :--- | :--- |
| $E_{u}^{L 2}$ | 7.507 | 7.507 | $E_{u}^{L 2}$ | 0.563 | 1.076 |
| $E_{u}^{R P D}$ | 0.00395 | 0.00395 | $E_{u}^{R P D}$ | 0.00234 | 0.01095 |
| $E_{v}^{L 2}$ | 29.5917 | 29.5917 | $E_{v}^{L 2}$ | 0.9572 | 0.94167 |
| $E_{v}^{R P D}$ | -0.01552 | -0.01416 | $E_{v}^{R P D}$ | 0.00674 | 0.00571 |

Figure 2.12: $L_{2}$ and RPD Error Measures for LR and BT Methods w/ $h=0.1$
surface behavior; whereas the RPB measure is bounded above and below. We see that the RPD value is less than $1 \%$ for Example 1 and approximately $1 \%$ for Example 2. To make our approximations more accurate, we can simply make $h$ closer to 0 . The results for both $h=0.01$ and $h=0.001$ are shown in the following figures.

| Example $1 \mathrm{w} / \mathrm{h}=0.01$ | LR Method | BT Method | Example $2 \mathrm{w} / \mathrm{h}=0.01$ | LR Method | BT Method |
| :---: | :--- | :--- | :---: | :--- | :--- |
| $E_{u}^{L 2}$ | 7.2458 | 7.2458 | $E_{u}^{L 2}$ | 0.54933 | 1.063 |
| $E_{u}^{R P D}$ | 0.00035 | 0.00037 | $E_{u}^{R P D}$ | 0.00024 | 0.00111 |
| $E_{v}^{L 2}$ | 28.9397 | 28.9397 | $E_{v}^{L 2}$ | 0.93937 | 0.9459 |
| $E_{v}^{R P D}$ | -0.00156 | -0.00143 | $E_{v}^{R P D}$ | 0.00068 | 0.00060 |

Figure 2.13: $L_{2}$ and RPD Error Measures for LR and BT Methods w/ $h=0.01$

| Example $1 \mathrm{w} / \mathrm{h}=0.001$ | LR Method | BT Method | Example 2 w/h $=0.001$ | LR Method | BT Method |
| :---: | :--- | :--- | :---: | :--- | :--- |
| $E_{u}^{L 2}$ | 7.2198 | 7.2198 | $E_{u}^{L 2}$ | 0.5480 | 1.0616 |
| $E_{u}^{R P D}$ | 0.0000413 | 0.0000413 | $E_{u}^{R P D}$ | 0.0000241 | 0.000111 |
| $E_{v}^{L 2}$ | 28.8747 | 28.8747 | $E_{v}^{L 2}$ | 0.9377 | 0.9463 |
| $E_{v}^{R P D}$ | -0.000156 | -0.000143 | $E_{v}^{R P D}$ | 0.0000679 | 0.0000598 |

Figure 2.14: $L_{2}$ and RPD Error Measures for LR and BT Methods w/ $h=0.001$

The question now remains is how we can improve the accuracy of our solution even more. As previously mentioned, the partial derivatives in (Eq 2.16) were approximated with a first-order forward-difference approximation, as shown in (Eq 2.17). It is intuitive that if we increase the order of accuracy for our approximation, then the accuracy of our methods should improve.

To begin, it should be necessary to convince that the methods discussed before are actually first-order. Recall that for an analytic function $f$ of two variables $x, y$, its first-order Taylor series is given by:

$$
\begin{equation*}
f\left(x+h_{x}, y+h_{y}\right)=f(x, y)+f_{x}(x, y) \cdot h_{x}+f_{y}(x, y) \cdot h_{y}+\mathcal{O}\left(h_{x}^{2}+h_{y}^{2}\right) \tag{2.20}
\end{equation*}
$$

To obtain an approximation for $f_{x}$, we set $h_{y}=0$ and will define $h:=h_{x}$. This gives:

$$
\begin{equation*}
f(x+h, y)=f(x, y)+f_{x}(x, y)+\mathcal{O}\left(h^{2}\right) \tag{2.21}
\end{equation*}
$$

Rearranging gives us:

$$
\begin{equation*}
f_{x}(x, y)=\frac{f(x+h, y)-f(x)}{h}+\mathcal{O}(h) \tag{2.22}
\end{equation*}
$$

An analogous relationship can be developed for $f_{y}$, which supports that ( Eq 2.8 ) and all immediate equations from it are first-order approximations. We will now take more terms of the Taylor series in hopes for better accuracy.

Now consider the second-order Taylor series:

$$
\begin{align*}
f\left(x+h_{x}, y+h_{y}\right) & =f(x, y)+f_{x}(x, y) \cdot h_{x}+f_{y}(x, y) \cdot h_{y}+ \\
& \frac{1}{2}\left[f_{x x}(x, y) \cdot h_{x}^{2}+2 \cdot f_{x y}(x, y) \cdot h_{x} h_{y}+f_{y y}(x, y) \cdot h_{y}^{2}\right]+\mathcal{O}\left(h_{x}^{3}+h_{y}^{3}\right) \tag{2.23}
\end{align*}
$$

Similar to before, we set $h_{y}=0$ and define $h:=h_{x}$ in hopes to obtain an approximation to $f_{x}$.

$$
\begin{equation*}
f(x+h, y)=f(x, y)+f_{x}(x, y) \cdot h+\frac{1}{2} f_{x x}(x, y) \cdot h^{2}+\mathcal{O}\left(h^{3}\right) \tag{2.24}
\end{equation*}
$$

Replacing $h$ with $-h$ in (Eq 2.24), one can get:

$$
\begin{equation*}
f(x-h, y)=f(x, y)-f_{x}(x, y) \cdot h+\frac{1}{2} f_{x x}(x, y) \cdot h^{2}-\mathcal{O}\left(h^{3}\right) \tag{2.25}
\end{equation*}
$$

Subtracting (Eq 2.25) from (Eq 2.24) we get:

$$
\begin{equation*}
f(x+h, y)-f(x-h, y)=2 \cdot f_{x}(x, y) \cdot h+2 \mathcal{O}\left(h^{3}\right) . \tag{2.26}
\end{equation*}
$$

Rearranging gives us:

$$
\begin{equation*}
f_{x}(x, y)=\frac{f(x+h, y)-f(x-h, y)}{2 h}+\mathcal{O}\left(h^{2}\right) \tag{2.27}
\end{equation*}
$$

An analogous relationship can be developed can be developed for $f_{y}$. As we began to develop the method in the previous section through (Eq 2.8), we will explain the process of the second-order method through the solution of $u$ via (Eq 2.6), and a similar process will be developed for the solution of $v$.

Using the second-order method developed in (Eq 2.27), once can approximate the solution $u$ via the equations:

$$
\begin{align*}
& u(x+h, y)=u(x-h, y)+2 h \cdot a(x, y)  \tag{2.28}\\
& u(x, y+h)=u(x, y-h)-2 h \cdot b(x, y)
\end{align*}
$$

Knowing the point at $x=x_{0}$ will allow for us to get the solution at $x=x_{2}$. We will use a first-order forward difference to obtain $x=x_{1}$, which will then allow us to obtain $x=x_{3}$, and hence all values in that corresponding direction. Applying the first-order forward difference method for the first step and continuing on with the second-order central difference method gives us the following results. As we see, the "second-order" method does not gain us that

| Example 1 w/h $=0.001$ <br> ${ }^{*} \mathbf{2}^{\text {nd }}$ Central Difference | LR Method | BT Method | Example 2 w/h $=0.001$ <br>  <br> ${ }^{*} \mathbf{2}^{\text {nd }}$ Central Difference | LR Method | BT Method |
| :---: | :--- | :--- | :--- | :--- | :--- |
| $E_{u}^{L 2}$ | 14.442 | 14.442 | $E_{u}^{L 2}$ | 1.8411 | 1.6412 |
| $E_{u}^{R P D}$ | 0.0000795 | 0.0000784 | $E_{u}^{R P D}$ | -0.000113 | 0.000168 |
| $E_{v}^{L 2}$ | 22.8295 | 22.8295 | $E_{v}^{L 2}$ | 1.1536 | 1.0212 |
| $E_{v}^{R P D}$ | -0.0000847 | -0.0000650 | $E_{v}^{R P D}$ | 0.0000692 | 0.0000196 |

Figure 2.15: Error Measures for Central-Difference-v1 LR/BT w/ $h=0.001$
much accuracy. As we see, the accuracy is not necessarily better. As we know from the realvalued functions with one independent variable, say $x$, the accuracy of the method depends (sometimes) on the nature of the exact solution in the direction of $d x$. Since we are solving throughout two independent variables $x, y$, the surface may be not be ideal throughout the planes $x=0$ (which is what we solve $u_{y}$ and $v_{y}$ through) or $y=0$ (the plane we solve $u_{x}$ and $v_{x}$ through). We see that the methods still produce adequate results provided that $h$ is sufficiently small, so it is not necessarily a "useless" method, rather an alternative. In regards to the numerical results, it sometimes gives us results that are better than the first-order forward difference method, and sometimes give us worse; but overall they are comparably equal. Future discussions and methods will investigate methods that are more sophisticate than the $L R$ and $B T$ approach.

Another issue that may be leading to the issue of "this isn't better than the first method" is the parallelism of the solution that the leap-frog method obtains. Notice that in the $x$ direction, the solution in the vector $x=x_{2 k}$ is independent of the solution $x=x_{2 k+1}, k \in \mathbb{Z}$. Therefore, one or possibly both of these solutions could be divergent from the actual solution, and hence may lead to error accumulation in the result.

A slight improvement can be made on this method in regards to the first step. First note that $u\left(x_{0}+h, y_{0}\right)=u\left(x_{0}, y_{0}\right)+h \cdot a\left(x_{0}, y_{0}\right)$ is a first-order approximation. We can obtain this by using two half-steps from the origin point.

$$
\begin{align*}
& u\left(x_{0}+0.5 h, y_{0}\right)=u\left(x_{0}, y_{0}\right)+\frac{h}{2} \cdot a\left(x_{0}, y_{0}\right) \\
& u\left(x_{0}+h, y_{0}\right)=u\left(x_{0}+0.5 h, y_{0}\right)+\frac{h}{2} \cdot a\left(x_{0}+0.5 h, y_{0}\right) \tag{2.29}
\end{align*}
$$

Combining these gives us:

$$
\begin{equation*}
u\left(x_{0}+h, y_{0}\right)=u\left(x_{0}, y_{0}\right)+\frac{h}{2} \cdot\left(a\left(x_{0}, y_{0}\right)+a\left(x_{0}+0.5 h, y_{0}\right)\right) \tag{2.30}
\end{equation*}
$$

This is the well-known mid-point method, which we also know to have order-2 accuracy. One can arrive at analogous equations for the remaining three differential equations. The results shown in Figure 2.16 are obtained from this slight modification. In comparison to

| Example $1 \mathrm{w} / \mathrm{h}=0.001$ <br> $2^{\text {nd }}$ Central Difference | LR Method | BT Method | Example 2 w/h $=0.001$ <br> $2^{\text {nd }}$ Central Difference | LR Method | BT Method |
| :---: | :--- | :--- | :--- | :--- | :--- |
| $E_{u}^{L 2}$ | 14.4414 | 14.4414 | $E_{u}^{L 2}$ | 1.8409 | 1.6408 |
| $E_{u}^{R P D}$ | 0.0000794 | 0.0000784 | $E_{u}^{R P D}$ | -0.000113 | 0.000167 |
| $E_{v}^{L 2}$ | 22.8295 | 22.8295 | $E_{v}^{L 2}$ | 1.1534 | 1.0211 |
| $E_{v}^{R P D}$ | -0.0000847 | -0.0000650 | $E_{v}^{R P D}$ | 0.0000692 | 0.0000196 |

Figure 2.16: Error Measures for Central-Difference-v2 LR/BT w/ $h=0.001$
the results shown in Figure 2.15, there is not a large difference in the results, but the the
difference is still in the positive direction.

This difference may be small due to the well-behaved nature of $u$ and $v$ and may be more beneficial of these solutions are more chaotic in space.

### 2.3 One-Dimensional Complex Spatial IVP's, where $f=f(z, \omega)$

The previous sections were devoted to the problem $\omega^{\prime}=f$, where $f$ is a function of $z$ only. We now focus on the more general problem, where $f$ is a function of both $z$ and the unknown function $\omega$; that is:

$$
\begin{equation*}
\frac{d \omega}{d z}=f(z, \omega), \quad \omega\left(z_{0}\right)=\omega_{0} \tag{2.31}
\end{equation*}
$$

To guide us on the intuition behind the method of discussion, we will begin with one of the most fundamental, but also important, examples:

$$
\begin{equation*}
\frac{d \omega}{d z}=\lambda \omega, \quad \omega(0)=1 \tag{2.32}
\end{equation*}
$$

We will refer to this example as "Example 3". We can decompose the left-hand side of (Eq 2.32) the same way as in (Eq 2.4) to give us:

$$
\begin{equation*}
\left(\frac{\partial u}{\partial z}+i \frac{\partial v}{\partial z}\right)=\lambda u+\lambda v i \tag{2.33}
\end{equation*}
$$

Again using (Eq 1.23), we have the system:

$$
\begin{align*}
& u_{x}+v_{y}=2 \cdot(\lambda u)  \tag{2.34}\\
& v_{x}-u_{y}=2 \cdot(\lambda v)
\end{align*}
$$

Since the functions $u, v$ satisfy C.R., we have the following set of equations that can allow for us to obtain our solution:

$$
\begin{array}{ll}
u_{x}=\lambda u & u_{y}=-\lambda v  \tag{2.35}\\
v_{x}=\lambda v & v_{y}=\lambda u
\end{array}
$$

In terms of numerical implementation, one must solve both $u$ and $v$ in a point-wise fashion simultaneously, whereas the before approach had the option of doing them independently. To compare our solution we will consider the computational domain $\mathcal{D}=[0,5] \times[0,5]$, with $\lambda=-1$. We then have the exact solution of (Eq 2.32) to be:

$$
\begin{align*}
\omega(z)=e^{\lambda z} & \rightarrow u(x, y)=\operatorname{Re}(\omega)=e^{\lambda x} \cos (\lambda y)  \tag{2.36}\\
& \rightarrow v(x, y)=\operatorname{Im}(\omega)=e^{\lambda x} \sin (\lambda y)
\end{align*}
$$

For the implementation, we will use the first-order difference method described in section 2.1 due to its simplicity and reliability that was discussed in 2.2. By comparison of Figures 2.17 with 2.18 , we see that this method produces the correct result. As a note, these graphs were constructed with $h=0.1$.


Figure 2.17: Real/Imaginary Parts for Exact Solution of $\omega^{\prime}=\lambda \omega$


Figure 2.18: Real/Imaginary Parts Numerical Solutions of $\omega^{\prime}=\lambda \omega$

We can numerically calculate the error as we had done before. As a note, the results are calculated for $h=0.005$ and not $h=0.001$ due to memory constraints; which again reminds why we leave a more accurate method in the future to consider.

| Example 3 w/h $=0.1$ <br> Forward Difference | LR Method | BT Method | Example 3 w/h $=0.005$ <br> Forward Difference | LR Method | BT Method |
| :---: | :--- | :--- | :--- | :--- | :--- |
| $E_{u}^{L 2}$ | 1.4407 | 1.4407 | $E_{u}^{L 2}$ | 1.2385 | 1.2385 |
| $E_{u}^{R D}$ | 0.010908 | 0.010908 | $E_{u}^{R P D}$ | 0.000515 | 0.000515 |
| $E_{v}^{L 2}$ | 1.8173 | 1.8173 | $E_{v}^{L 2}$ | 1.5608 | 1.5608 |
| $E_{v}^{R P D}$ | -0.0069541 | -0.0069541 | $E_{v}^{R P D}$ | -0.0003708 | -0.0003708 |

Figure 2.19: Error Measures for Numerical Solutions w/ $h=0.1$ and $h=0.005$

Up to whatever level of algebraic tools, functions, cleverness and overall patience one may have at their disposal, the real and imaginary parts of $f$ for ( Eq 2.31 ) are always able to be obtained manually. For swiftness, we would like to develop a method that obtains the real and imaginary parts of $f$ (which we will denote as $a$ and $b$ respectively) in an implicit and/or point-wise fashion.

Rewriting (Eq 2.31) into its corresponding real and imaginary components together with the Wirtinger representation of $\frac{\partial}{\partial z}$, we obtain:

$$
\begin{equation*}
\left(\frac{\partial}{\partial x}-i \frac{\partial}{\partial y}\right)(u+i v)=2 \cdot f(x, y, u, v) \tag{2.37}
\end{equation*}
$$

Applying our differential operator and equating real and imaginary parts of the equation, we obtain:

$$
\begin{align*}
& u_{x}+v_{y}=2 \cdot \operatorname{Re}(f):=2 \cdot a(x, y, u, v)  \tag{2.38}\\
& v_{x}-u_{y}=2 \cdot \operatorname{Im}(f):=2 \cdot b(x, y, u, v)
\end{align*}
$$

This will work in the intuitive fashion. To demonstrate, consider the equation $u_{x}=a$. Using a first-order scheme, starting with the initial condition, we have:

$$
\begin{equation*}
u\left(x_{0}+h, y\right)=u\left(x_{0}, y_{0}\right)+h \cdot \operatorname{Re}\left\{f\left(x_{0}, y_{0}, u\left(x_{0}, y_{0}\right), v\left(x_{0}, y_{0}\right)\right)\right\} \tag{2.39}
\end{equation*}
$$

Since the $\operatorname{Re}\{f\}$ and $\operatorname{Im}\{f\}$ are always known at values $(x, y)$ that fall behind our left-hand side $(x, y)$ location, the solution method is easy to generalize.

To demonstrate, consider our fourth example given by:

$$
\begin{equation*}
\frac{d \omega}{d z}=z-\omega, \quad \omega(0)=1+3 i \tag{2.40}
\end{equation*}
$$

The exact solution can be found to be:

$$
\omega(z)=(2+3 i) \cdot e^{-z}+z-1 \rightarrow\left\{\begin{array}{l}
u(x, y)=\operatorname{Re}(\omega)=e^{-x}(2 \cos (y)+3 \sin (y))+x-1  \tag{2.41}\\
v(x, y)=\operatorname{Im}(\omega)=e^{-x}(-2 \sin (y)+3 \cos (y))+y
\end{array}\right.
$$

Computing on $\mathcal{D}=[0,3] \times[0,3]$ gives us the following results.


Figure 2.20: Real/Imaginary Parts for Exact Solution of $\omega^{\prime}=z-\omega$

| Example 4 w/h $=0.1$ <br> Forward Difference | LR Method | BT Method | Example 4 w/h $=0.001$ <br> Forward Difference | LR Method | BT Method |
| :---: | :--- | :--- | :--- | :--- | :--- |
| $E_{u}^{L 2}$ | 1.7597 | 2.5588 | $E_{u}^{L 2}$ | 1.692 | 2.477 |
| $E_{u}^{R P D}$ | -0.031537 | 0.00075117 | $E_{u}^{R P D}$ | -0.0003195 | 0.00007943 |
| $E_{v}^{L 2}$ | 3.3804 | 4.7509 | $E_{v}^{L 2}$ | 3.1634 | 4.646 |
| $E_{v}^{R P D}$ | 0.010608 | 0.070509 | $E_{v}^{R P D}$ | 0.0001139 | 0.0006996 |

Figure 2.21: Error Measures for Numerical Solutions w/ $h=0.1$ and $h=0.001$

For the last example, consider the non-linear IVP given by:

$$
\begin{equation*}
\frac{d \omega}{d z}=\frac{1+\omega^{2}}{1+z}, \quad \omega(-7-7 i)=3-7 i \tag{2.42}
\end{equation*}
$$

One can find that the exact solution is given by:

$$
\begin{align*}
\omega(z) & =\tan (c+\ln (1+z))  \tag{2.43}\\
c & =\arctan \left(\omega_{0}\right)-\ln \left(1+z_{0}\right)
\end{align*}
$$

For all points $(x, y)$, we can calculate the real and imaginary parts of $\omega$ in a point-wise fashion since we do not have the explicit representation. Computing the solution on $\mathcal{D}=$ $[-7,-1.5] \times[-7,-1.5]$ gives us the following results.



Figure 2.22: Real/Imaginary Parts for Exact Solution of $\omega^{\prime}=\frac{1+\omega^{2}}{1+z}$

| Example 5 w/h = 0.1 <br> Forward Difference | LR Method | BT Method | Example 5 w/h = 0.005 <br> Forward Difference | LR Method | BT Method |
| :---: | :--- | :--- | :--- | :--- | :--- |
| $E_{u}^{L 2}$ | 3.3479 | 8.5191 | $E_{u}^{L 2}$ | 2.9253 | 6.8067 |
| $E_{u}^{R P D}$ | 0.024352 | -0.009387 | $E_{u}^{R P D}$ | 0.0011511 | -0.0006809 |
| $E_{v}^{L 2}$ | 3.7796 | 9.2816 | $E_{v}^{L 2}$ | 3.4322 | 7.7093 |
| $E_{v}^{R P D}$ | -0.028524 | -0.098694 | $E_{v}^{R P D}$ | -0.0014895 | -0.0046852 |

Figure 2.23: Error Measures for Numerical Solutions w/ $h=0.1$ and $h=0.001$

Now that we have a method that solves the general first-order initial value problem $\omega^{\prime}=$ $f(z, \omega)$ in which gives us adequate results for $u$ and $v$, this implies that we can get an understanding of the actual complex surface which we can view as a Riemann Surface. One can see that fourth numerical Riemann surfaces are practically equal, whereas the LR Riemann surface of example five is closer to the exact. Intuitively, the differences between the exact Riemann surface and numerical surfaces converge to 0 as $h$ gets closer to 0 .


Figure 2.24: Numerical LR/BT and Exact Riemann Surface for $\omega^{\prime}=z-\omega$


Figure 2.25: Numerical LR/BT and Exact Riemann Surface for $\omega^{\prime}=\frac{1+\omega^{2}}{1+z}$

### 2.4 Laplace Transformation Solutions to Complex BVP's

One of the methods that I, throughout my individual development, personally found to be the most interesting is the application of using Laplace transforms to solve differential
equations. This approach is guided by the fact that the Laplace transform transforms a differential equation into an algebraic equation for which we can solve for the Laplace transform of our unknown solution, and then invert the Laplace transform to obtain the solution to the original differential equation. To guide our motivation into the complex realm, consider the basic real-variable IVP:

$$
y^{\prime}(x)=2 x, \quad y(0)=3
$$

Taking the 1-dimensional Laplace transform of both sides gives us:

$$
\begin{aligned}
\int_{0}^{\infty} y^{\prime}(x) e^{-x \xi} d x=\int_{0}^{\infty}(2 x) e^{-x \xi} d x & \rightarrow \xi \cdot \mathcal{L}\{y\}-y(0)=2 \cdot \frac{1}{\xi^{2}} \\
& \rightarrow \mathcal{L}\{y\}=\left(\frac{2}{\xi^{3}}\right)+3 \cdot\left(\frac{1}{\xi}\right)
\end{aligned}
$$

One can verify, from the definition of the transform, that:

$$
\begin{aligned}
y & =\mathcal{L}^{-1}\left\{\frac{2}{\xi^{3}}\right\}+3 \mathcal{L}^{-1}\left\{\frac{1}{\xi}\right\} \\
& =x^{2}+3
\end{aligned}
$$

Instead of solving the initial value problem, we will instead solve a related boundary value problem (reasons to be discussed in the upcoming paragraphs).

$$
\begin{align*}
\omega^{\prime}(z) & =2 z  \tag{2.44}\\
u(0, y) & =-y^{2}+3, \quad u(x, 0)=x^{2}+3, \quad v(0, y)=v(x, 0)=0 .
\end{align*}
$$

It is again important to recall that we are not simply replacing function names and renaming variable names, rather we are solving an equation of an arbitrary complex input, $z$, with an unknown complex output, $\omega$. We will approach this problem in a similar manner to the numerical approach; we will assume both $\omega$ and $z$ in their two-dimensional partitions ( $u, v$ )
and $(x, y)$, respectively. Therefore, we when taking the Laplace transform of a complex variable function, we will define it as the two-dimensional real-variable Laplace transform in terms of $x$ and $y$. For our particular problem, we desire the Laplace transform of $z$ (the constant is trivially approached). We can decompose this into the traditional way to give us:

$$
\begin{equation*}
\mathcal{L}\{z\}=\mathcal{L}\{x\}+i \cdot \mathcal{L}\{y\} . \tag{2.45}
\end{equation*}
$$

Focusing on the real part only (since the imaginary part can be constructed in parallel), we have:

$$
\begin{align*}
\mathcal{L}\{x\} & =\int_{0}^{\infty} \int_{0}^{\infty}(x) \cdot e^{-x \xi_{1}-y \xi_{2}} d x d y \\
& =\int_{0}^{\infty} e^{-y \xi_{2}}\left(\int_{0}^{\infty} x \cdot e^{-x \xi_{1}} d x\right) d y \\
& =\frac{1}{\xi_{1}^{2}} \cdot \int_{0}^{\infty} e^{-y \xi_{2}} d y  \tag{2.46}\\
& =\frac{1}{\xi_{1}^{2} \cdot \xi_{2}}
\end{align*}
$$

Multiplying both sides of (Eq 2.45) by 2 and applying (Eq 2.46) (and its analogue) to the real and imaginary parts of (Eq 2.45), we obtain the Laplace transform of the right-hand side of (Eq 2.44) to be:

$$
\begin{equation*}
\mathcal{L}\{2 z\}=\frac{2}{\xi_{1}^{2} \xi_{2}}+\frac{2 i}{\xi_{1} \xi_{2}^{2}} . \tag{2.47}
\end{equation*}
$$

The portion of interest is, of course, the left-hand side of the original equation, since it applies to every initial-value problem in question. We start with the guiding identity:

$$
\frac{d \omega}{d z}=\frac{\partial u}{\partial z}+i \frac{\partial v}{\partial z}=\frac{1}{2}\left(\frac{\partial u}{\partial x}-i \frac{\partial u}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}-i \frac{\partial v}{\partial y}\right)
$$

Upon simplification and applying the Cauchy-Riemann equations, we obtain the parallel equations:

$$
\begin{align*}
w^{\prime} & =u_{x}-i u_{y}  \tag{2.48}\\
w^{\prime} & =v_{y}+i v_{x}
\end{align*}
$$

We only take these two equations into consideration, because taking the Laplace transform of each will give us an equation containing only the Laplace transform of the real and imaginary parts of the exact solution $\omega$, which then only requires knowledge of the inverse Laplace transform.

To derive a representation for $\mathcal{L}\left\{\omega^{\prime}\right\}$, we will first analyze the Laplace transform of $u_{x}$ and take the other three as analogous, and similarly derivable, identities. Using the definition of the Laplace transform, Fubini's theorem, and integration by parts, we have:

$$
\begin{align*}
\mathcal{L}\left\{u_{x}\right\} & =\int_{0}^{\infty} \int_{0}^{\infty} u_{x}(x, y) e^{-x \xi_{1}-y \xi_{2}} d x d y \\
& =\int_{0}^{\infty} e^{-y \xi_{2}}\left(\int_{0}^{\infty} u_{x}(x, y) e^{-x \xi_{1}} d x\right) d y  \tag{2.49}\\
& =\int_{0}^{\infty} e^{-y \xi_{2}}\left(\left(\left.u(x, y) e^{-x \xi_{1}}\right|_{x \rightarrow 0} ^{x \rightarrow \infty}-\int_{0}^{\infty}\left(-\xi_{1}\right) u(x, y) e^{-x \xi_{1}} d x\right) d y\right. \\
& =-\int_{0}^{\infty} u(0, y) e^{-y \xi_{2}} d y+\xi_{1} \int_{0}^{\infty} \int_{0}^{\infty} u(x, y) e^{-x \xi_{1}-y \xi_{2}} d x d y
\end{align*}
$$

Rewriting this result in a more condensed manner, we have the identity:

$$
\begin{equation*}
\mathcal{L}\left\{u_{x}\right\}=\xi_{1} \cdot \mathcal{L}\{u(x, y)\}-\mathcal{L}\{u(0, y)\} . \tag{2.50}
\end{equation*}
$$

Similarly, we have the following identity (and similar for $v$ ):

$$
\begin{equation*}
\mathcal{L}\left\{u_{y}\right\}=\xi_{2} \cdot \mathcal{L}\{u(x, y)\}-\mathcal{L}\{u(x, 0)\} \tag{2.51}
\end{equation*}
$$

To understand how these equations are implemented, let us continue our solution of (Eq 2.44). Using the $u$ representation of $\omega^{\prime}(\operatorname{Eq} 2.48)$ and then taking the Laplace transform of both sides of (Eq 2.44), we have:

$$
\begin{equation*}
\xi_{1} \mathcal{L}\{u\}-\mathcal{L}\{u(0, y)\}-i \xi_{2} \mathcal{L}\{u\}+i \mathcal{L}\{u(x, 0)\}=\frac{2}{\xi_{1}^{2} \xi_{2}}+\frac{2 i}{\xi_{1} \xi_{2}^{2}} \tag{2.52}
\end{equation*}
$$

Equating real and imaginary parts, we obtain the Laplace-transform system:

$$
\begin{align*}
\mathcal{L}\{u\} & =\frac{1}{\xi_{1}} \mathcal{L}\{u(0, y)\}+\frac{2}{\xi_{1}^{3} \xi_{2}}  \tag{2.53}\\
\mathcal{L}\{u\} & =\frac{1}{\xi_{2}} \mathcal{L}\{u(x, 0)\}-\frac{2}{\xi_{1} \xi_{2}^{3}} .
\end{align*}
$$

Here one can see why I chose to inscribe boundary conditions instead of an initial-value point, as the problems difficulty then increases, which we approach in the following section. One may note that we have two equations (from Eq 2.53 ) that will allow us to obtain the solution of $u$, therefore for the boundary-value problem to be well defined, we only need one of the boundary conditions. Since the existence and uniqueness theorem guarantees uniqueness, we will illustrate that both of these equations (Eq 2.53) give us the same solution. Since the Laplace transforms of the boundary conditions are one-dimensional, we can use the traditional Laplace-transform identities used in earlier differential equations courses; therefore:

$$
\begin{align*}
\mathcal{L}\{u(0, y)\} & =\mathcal{L}\left\{-y^{2}+3\right\} \\
& =-\frac{2}{\xi_{2}^{3}}+\frac{3}{\xi_{2}} .  \tag{2.54}\\
\mathcal{L}\{u(x, 0)\} & =\mathcal{L}\left\{x^{2}+3\right\} \\
& =\frac{2}{\xi_{1}^{3}}+\frac{3}{\xi_{1}} .
\end{align*}
$$

Substituting the Laplace-transform of the boundary conditions into our Laplace system, one obtains the equation (for both) to be:

$$
\begin{equation*}
\mathcal{L}\{u\}=\frac{2}{\xi_{1}^{3} \xi_{2}}-\frac{2}{\xi_{1} \xi_{2}^{3}}+\frac{3}{\xi_{1} \xi_{2}} . \tag{2.55}
\end{equation*}
$$

Note here that (Eq 2.55) is symmetric with respect to both variables (aside from signs that are due to the $i$ term); this allows us to see that it doesn't matter which variable we focus on. Taking the two-dimensional inverse Laplace transform of both sides of (Eq 2.55), we have:

$$
\begin{align*}
u(x, y) & =\mathcal{L}^{-1}\left\{\frac{2}{\xi_{1}^{3} \xi_{2}}\right\}-\mathcal{L}^{-1}\left\{\frac{2}{\xi_{1} \xi_{2}^{3}}\right\}+\mathcal{L}^{-1}\left\{\frac{3}{\xi_{1} \xi_{2}}\right\}  \tag{2.56}\\
& =x^{2}-y^{2}+3
\end{align*}
$$

To find the imaginary part of $\omega$, we will take the $v$ representation of (Eq 2.48) apply the Laplace transform and proceed similarly. We will then have:

$$
\begin{align*}
\mathcal{L}\left\{\omega^{\prime}\right\} & =\mathcal{L}\left\{v_{y}+i v_{x}\right\} \\
& =\xi_{2} \mathcal{L}\{v\}-\mathcal{L}\{v(x, 0)\}+i \xi_{1} \mathcal{L}\{v\}-i \mathcal{L}\{v(0, y)\}  \tag{2.57}\\
& \equiv \frac{2}{\xi_{1}^{2} \xi_{2}}+\frac{2 i}{\xi_{1} \xi_{2}^{2}} .
\end{align*}
$$

Isolating and equating real and imaginary parts will give rise to a corresponding Laplace system for $v$.

$$
\begin{align*}
& \mathcal{L}\{v\}=\frac{1}{\xi_{2}} \mathcal{L}\{v(x, 0)\}+\frac{2}{\xi_{1}^{2} \xi_{2}^{2}}  \tag{2.58}\\
& \mathcal{L}\{v\}=\frac{1}{\xi_{1}} \mathcal{L}\{v(0, y)\}+\frac{2}{\xi_{1}^{2} \xi_{2}^{2}} .
\end{align*}
$$

It is given that the boundary conditions for $v$ are both 0 , hence it is a trivial calculation that the Laplace transform of the boundary conditions will vanish as well. Hence we can see that
this results in the single equation:

$$
\begin{equation*}
\mathcal{L}\{v\}=\frac{2}{\xi_{1}^{2} \xi_{2}^{2}} \tag{2.59}
\end{equation*}
$$

One can easily verify using Fubinis theorem that:

$$
\begin{align*}
v(x, y) & =2 \cdot \mathcal{L}^{-1}\left(\frac{1}{\xi_{1}^{2} \xi_{2}^{2}}\right)  \tag{2.60}\\
& =2 x y
\end{align*}
$$

This gives us the exact solution to be:

$$
\omega(x+i y)=\left(x^{2}-y^{2}+3\right)+i(2 x y)
$$

which we have already verified. It is useful to know an identity that discusses the complex transform of a complex-variable polynomial; that is, an analogous version of the real-variable identity:

$$
\mathcal{L}\left\{x^{n}\right\}=\frac{n!}{\xi^{n+1}} .
$$

We begin first by recalling the binomial theorem, i.e.:

$$
\begin{equation*}
(x+i y)^{n}=\sum_{k=0}^{n}\binom{n}{k} x^{n-k}(i y)^{k} . \tag{2.61}
\end{equation*}
$$

Implementing the definition of the Laplace transform and performing the necessary manipulations and using (Eq 2.61), we get:

$$
\begin{align*}
\mathcal{L}\left\{z^{n}\right\} & =\int_{0}^{\infty} \int_{0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} x^{n-k}(i y)^{k}\right) \cdot e^{-x \xi_{1}-y \xi_{2}} d x d y \\
& =\sum_{k=0}^{n}\left[\binom{n}{k} i^{k}\left(\int_{0}^{\infty} x^{n-k} e^{-x \xi_{1}}\left(\int_{0}^{\infty} y^{k} e^{-y \xi_{2}} d y\right) d x\right)\right]  \tag{2.62}\\
& =\sum_{k=0}^{n}\left[\binom{n}{k} i^{k}\left(\int_{0}^{\infty} x^{n-k} \cdot e^{-x \xi_{1}} d x\right) \cdot \frac{k!}{\xi_{2}^{k+1}}\right] \\
& =\sum_{k=0}^{n}\binom{n}{k} i^{k} \cdot \frac{k!\cdot(n-k)!}{\xi_{1}^{n-k+1} \xi_{2}^{k+1}} .
\end{align*}
$$

We can simplify the combinatorial with the numerator constants to obtain the simplified identity for the Laplace transform of complex polynomials:

$$
\begin{equation*}
\mathcal{L}\left\{z^{n}\right\}=n!\cdot \sum_{k=0}^{n} \frac{i^{k}}{\xi_{1}^{n-k+1} \xi_{2}^{k+1}} \tag{2.63}
\end{equation*}
$$

Another useful two-dimensional identity that can be proven almost identically as was demonstrated in (Eq 2.62) is the following:

$$
\begin{equation*}
\mathcal{L}\left\{x^{p} \cdot y^{q}\right\}=\frac{p!q!}{\xi_{1}^{p+1} \xi_{2}^{q+1}} \tag{2.64}
\end{equation*}
$$

To reiterate the process of solving complex differential equations with the Laplace transform and also introduce a new topic of discussion, consider the following general differential equation:

$$
\begin{equation*}
\omega^{\prime}=f(z, \omega) \tag{2.65}
\end{equation*}
$$

To avoid a lack of motivation for the process, I intentionally will not state a boundary/initial condition. To solve, we first take the two dimensional Laplace transform of both sides in
order to obtain the decomposition functions $u$ and $v$ by using (Eq 2.48).

$$
\begin{align*}
\mathcal{L}\left\{u_{x}-i u_{y}\right\} & =\xi_{1} \mathcal{L}\{u\}-\mathcal{L}\{u(0, y)\}+i\left(-\xi_{2} \mathcal{L}\{u\}+\mathcal{L}\{u(x, 0)\}=\mathcal{L}\{f\}\right.  \tag{2.66}\\
\mathcal{L}\left\{v_{y}+i v_{x}\right\} & =\xi_{2} \mathcal{L}\{v\}-\mathcal{L}\{v(x, 0)\}+i\left(\xi_{1} \mathcal{L}\{v\}-\mathcal{L}\{v(0, y)\}=\mathcal{L}\{f\}\right.
\end{align*}
$$

Equating real and imaginary parts gives the following two pairs of systems:

$$
\begin{align*}
\mathcal{L}\{u\} & =\frac{1}{\xi_{1}}[\mathcal{L}\{u(0, y)\}+\operatorname{Re}\{\mathcal{L}\{f\}\}]  \tag{2.67}\\
\mathcal{L}\{u\} & =\frac{1}{\xi_{2}}[\mathcal{L}\{u(x, 0)\}-\operatorname{Im}\{\mathcal{L}\{f\}\}] \\
\mathcal{L}\{v\} & =\frac{1}{\xi_{2}}[\mathcal{L}\{v(x, 0)\}+\operatorname{Re}\{\mathcal{L}\{f\}\}]  \tag{2.68}\\
\mathcal{L}\{v\} & =\frac{1}{\xi_{1}}[\mathcal{L}\{v(0, y)\}+\operatorname{Im}\{\mathcal{L}\{f\}\}]
\end{align*}
$$

The first observation that one could make is that in order to obtain $u$ and $v$, you only need to know either the real OR imaginary part of the Laplace transform of $f$. This could be beneficial for calculations or numerical consistency measures. The second observation is that the solution is uniquely determined from only one boundary of $\mathcal{R}$, and not both. For instances, you can solve (Eq 1.67.1) and (Eq 1.68.1) and only need the $y$ boundary in order to calculate both $u$ and $v$. We will discuss this in the conclusion remarks of this chapter in more detail, but for now we just take it as an observation.

With that being mentioned, it is important to note that one cannot just randomly assign different boundary conditions in the $x$ and $y$ directions, as the resulting solutions may be completely different. To guide us in this investigation, consider the following differential equation:

$$
\omega^{\prime}=z^{2}-2 z+5 .
$$

I will leave out the prescribed boundary conditions for the time being. Before implementing ( Eq 2.67 ) and ( Eq 2.68 ), we first calculate the Laplace transform of $f$. Implementing ( Eq 2.63):

$$
\begin{align*}
\mathcal{L}\left\{z^{2}-2 z+5\right\} & =2!\left(\frac{i^{0}}{\xi_{1}^{3} \xi_{2}^{1}}+\frac{i^{1}}{\xi_{1}^{2} \xi_{2}^{2}}+\frac{i^{2}}{\xi_{1}^{1} \xi_{2}^{3}}\right)-1!\left(\frac{i^{0}}{\xi_{1}^{2} \xi_{2}^{1}}+\frac{i^{1}}{\xi_{1}^{1} \xi_{2}^{2}}\right) \cdot 2+0!\left(\frac{i^{0}}{\xi_{1}^{1} \xi_{2}^{1}}\right) \cdot 5 \\
& =\left(\frac{2}{\xi_{1}^{3} \xi_{2}}-\frac{2}{\xi_{1} \xi_{2}^{3}}-\frac{2}{\xi_{1}^{2} \xi_{2}}+\frac{5}{\xi_{1} \xi_{2}}\right)+i\left(\frac{2}{\xi_{1}^{2} \xi_{2}^{2}}-\frac{2}{\xi_{1} \xi_{2}^{2}}\right) . \tag{2.69}
\end{align*}
$$

Now let us assume that the boundary conditions for the equation of interest to be:

$$
\begin{align*}
\omega^{\prime} & =z^{2}-2 z+5 \\
u(0, y) & =y^{2}+2 \\
u(x, 0) & =\frac{1}{3} x^{3}-x^{2}+5 x+2  \tag{2.70}\\
v(0, y) & =-\frac{1}{3} y^{3}+5 y \\
v(x, 0) & =0
\end{align*}
$$

If the premise equation is (Eq 2.70) then the (one-dimensional) Laplace transform of the boundary conditions are:

$$
\begin{align*}
\mathcal{L}\{u(0, y)\} & =\frac{2}{\xi_{2}^{3}}+\frac{2}{\xi_{2}} \\
\mathcal{L}\{u(x, 0)\} & =\frac{2}{\xi_{1}^{4}}-\frac{2}{\xi_{1}^{3}}+\frac{5}{\xi_{1}^{2}}+\frac{2}{\xi_{1}}  \tag{2.71}\\
\mathcal{L}\{v(0, y)\} & =-\frac{2}{\xi_{2}^{4}}+\frac{5}{\xi_{2}^{2}} \\
\mathcal{L}\{v(x, 0)\} & =0
\end{align*}
$$

Substituting into (both pairs of) (Eq 2.67) and (Eq 2.68) to obtain (equivalent) expressions for $\mathcal{L}\{u\}$ and $\mathcal{L}\{v\}$, we get:

$$
\begin{gather*}
\mathcal{L}\{u\}=\frac{2}{\xi_{1}^{4} \xi_{2}}-\frac{2}{\xi_{1}^{3} \xi_{2}}+\frac{5}{\xi_{1}^{2} \xi_{2}}+\frac{2}{\xi_{1} \xi_{2}}-\frac{2}{\xi_{1}^{2} \xi_{2}^{3}}+\frac{2}{\xi_{1} \xi_{2}^{3}},  \tag{2.72}\\
\rightarrow \quad u(x, y)=\frac{1}{3} x^{3}+y^{2}-x y^{2}-x^{2}+5 x+2 . \\
\mathcal{L}\{v\}=-\frac{2}{\xi_{1} \xi_{2}^{4}}+\frac{5}{\xi_{1} \xi_{2}^{2}}+\frac{2}{\xi_{1}^{3} \xi_{2}^{2}}-\frac{2}{\xi_{1}^{2} \xi_{2}^{2}},  \tag{2.73}\\
\rightarrow \quad v(x, y)=-\frac{1}{3} y^{3}+5 y+x^{2} y-2 x y .
\end{gather*}
$$

Directly separating the complex solution of $\omega(z)=\frac{1}{3} z^{3}-z^{2}+5 z+2$ into its real and imaginary parts, we can verify that these are indeed correct.

It is important to mention that the process to solve an boundary value problem of the form $\omega^{\prime}=f(z)$ can be solved by the following sequence of steps:

- Find the 2-dimensional Laplace transform of $f(z)$.
- Find the 1-dimensional Laplace transform of the boundary conditions, which can then be used in the 2-D transform.
- (Optional) Verify that the paired systems of (Eq 2.67) and (Eq 2.68) are equal.
- Solve equations (2.67) and (2.68) by inverting the Laplace transform.

Note that (Eq 2.67) and (Eq 2.68) can only be used directly if $f=f(z)$, otherwise some modifications need to be made. As an example, consider:

$$
\begin{equation*}
\omega^{\prime}=z+\omega . \tag{2.74}
\end{equation*}
$$

Since the real-variable analogue of this equation involves an exponential, it is possible that the complex-variable version also does. For the 2-dimensional Laplace transform $f(z)=e^{z}$, we have:

$$
\begin{align*}
\mathcal{L}\left\{e^{x} \cdot e^{i y}\right\} & =\int_{0}^{\infty} \int_{0}^{\infty}\left(e^{x} e^{i y}\right) \cdot e^{-x \xi_{1}-y \xi_{2}} d x d y \\
& =\int_{0}^{\infty} e^{x} e^{-x \xi_{1}}\left(\int_{0}^{\infty} e^{i y} e^{-y \xi_{2}} d y\right) d x  \tag{2.75}\\
& =\int_{0}^{\infty} e^{x} e^{-x \xi_{1}}\left(\frac{i+\xi_{2}}{1+\xi_{2}^{2}}\right) d x \\
& =\frac{1}{\xi_{1}-1} \cdot \frac{i+\xi_{2}}{1+\xi_{2}^{2}} .
\end{align*}
$$

One can verify this using Euler's identity if they desire:

$$
\begin{aligned}
\mathcal{L}\left\{e^{x} \cdot e^{i y}\right\} & =\mathcal{L}\left\{e^{x} \cos (y)\right\}+i \cdot \mathcal{L}\left\{e^{x} \sin (y)\right\} \\
& =\int_{0}^{\infty} e^{x} e^{-x \xi_{1}}\left(\int_{0}^{\infty} \cos (y) e^{-y \xi_{2}} d y\right) d x+i \cdot \int_{0}^{\infty} e^{x} e^{-x \xi_{1}}\left(\int_{0}^{\infty} \sin (y) e^{-y \xi_{2}} d y\right) d x \\
& =\int_{0}^{\infty} e^{x} e^{-x \xi_{1}}\left(\frac{\xi_{2}}{1+\xi_{2}^{2}}\right) d x+i \int_{0}^{\infty} e^{x} e^{-x \xi_{1}}\left(\frac{1}{1+\xi_{2}^{2}}\right) d x \\
& =\frac{1}{\xi_{1}-1} \cdot \frac{i+\xi_{2}}{1+\xi_{2}^{2}} .
\end{aligned}
$$

Starting by taking the Laplace transform of both sides of (Eq 2.74) and splitting versions via (Eq 2.48), we obtain the system:

$$
\begin{align*}
\mathcal{L}\left\{u_{x}-i u_{y}\right\} & =\frac{1}{\xi_{1}^{2} \xi_{2}}+\frac{i}{\xi_{1} \xi_{2}^{2}}+\mathcal{L}\{u\}+i \mathcal{L}\{v\} \\
\mathcal{L}\left\{v_{y}+i v_{x}\right\} & =\frac{1}{\xi_{1}^{2} \xi_{2}}+\frac{i}{\xi_{1} \xi_{2}^{2}}+\mathcal{L}\{u\}+i \mathcal{L}\{v\} \tag{2.76}
\end{align*}
$$

Implementing (Eq 2.50) and (Eq 2.51) and equating real and imaginary parts gives:

$$
\begin{align*}
& \xi_{1} \mathcal{L}\{u\}-\mathcal{L}\{u(0, y)\}=\frac{1}{\xi_{1}^{2} \xi_{2}}+\mathcal{L}\{u\}  \tag{2.77}\\
& -\xi_{2} \mathcal{L}\{u\}+\mathcal{L}\{u(x, 0)\}=\frac{1}{\xi_{1} \xi_{2}^{2}}+\mathcal{L}\{v\} \\
& \xi_{2} \mathcal{L}\{v\}-\mathcal{L}\{v(x, 0)\}=\frac{1}{\xi_{1}^{2} \xi_{2}}+\mathcal{L}\{u\} \\
& \xi_{1} \mathcal{L}\{v\}-\mathcal{L}\{v(0, y)\}=\frac{1}{\xi_{1} \xi_{2}^{2}}+\mathcal{L}\{v\} \tag{2.78}
\end{align*}
$$

One should notice that (Eq 2.77.2) and (Eq 2.78.1) require knowledge of both $u$ and the unknown $v$ (or vice versa). Due to the uniqueness of Laplace transforms, we can instead just focus on (Eq 2.77.1) and (Eq 2.78.2). To solve these equations, we only need to know the boundary conditions $u(0, y)$ and $v(0, y)$. Let us now re-assume that the given boundary value problem was given to be:

$$
\begin{align*}
\omega^{\prime} & =z+\omega \\
u(0, y) & =3 \cos (y)-3 \sin (y)-1  \tag{2.79}\\
v(0, y) & =3 \cos (y)+3 \sin (y)-y
\end{align*}
$$

Taking the Laplace transform of the boundary condition gives:

$$
\begin{aligned}
& \mathcal{L}\{u(0, y)\}=\frac{3 \xi_{2}}{1+\xi_{2}^{2}}-\frac{3}{1+\xi_{2}^{2}}-\frac{1}{\xi_{2}} \\
& \mathcal{L}\{v(0, y)\}=\frac{3 \xi_{2}}{1+\xi_{2}^{2}}+\frac{3}{1+\xi_{2}^{2}}-\frac{1}{\xi_{2}^{2}}
\end{aligned}
$$

Substituting into ( Eq 2.77 .1 ) and ( Eq 2.78 .2 ) and using algebraic rearrangements when necessary (namely to the last two fractions in sequence), we obtain:

$$
\begin{align*}
& \left(\xi_{1}-1\right) \cdot \mathcal{L}\{u\}=\frac{3 \xi_{2}}{1+\xi_{2}^{2}}-\frac{3}{1+\xi_{2}^{2}}-\frac{1}{\xi_{2}}+\frac{1}{\xi_{1}^{2} \xi_{2}} \\
& \rightarrow \quad \mathcal{L}\{u\}=\frac{3 \xi_{2}}{\left(\xi_{1}-1\right)\left(1+\xi_{2}^{2}\right)}-\frac{3}{\left(\xi_{1}-1\right)\left(1+\xi_{2}^{2}\right)}-\frac{1}{\xi_{1}^{2} \xi_{2}}-\frac{1}{\xi_{1} \xi_{2}}  \tag{2.80}\\
& \rightarrow \quad u(x, y)=3 e^{x} \cos (y)-3 e^{x} \sin (y)-x-1, \\
& \left(\xi_{1}-1\right) \cdot \mathcal{L}\{v\}=\frac{3 \xi_{2}}{1+\xi_{2}^{2}}+\frac{3}{1+\xi_{2}^{2}}-\frac{1}{\xi_{2}^{2}}+\frac{1}{\xi_{1} \xi_{2}^{2}} \\
& \quad \rightarrow \quad \mathcal{L}\{v\}=\frac{3 \xi_{2}}{\left(\xi_{1}-1\right)\left(1+\xi_{2}^{2}\right)}+\frac{3}{\left(\xi_{1}-1\right)\left(1+\xi_{2}^{2}\right)}-\frac{1}{\xi_{1} \xi_{2}^{2}}  \tag{2.81}\\
& \rightarrow \quad v(x, y)=3 e^{x} \cos (y)+3 e^{x} \sin (y)-y,
\end{align*}
$$

which one can verify to be the real and imaginary parts of:

$$
\omega(z)=(3+3 i) e^{z}-z-1
$$

### 2.5 The Connection Between Boundaries and Initial Conditions

As discussed in the previous section, in order to implement Laplace transforms to solve complex-variable (or even two-dimensional real-variable) differential equations, the boundary conditions for $u$ and $v$ must be, at least partially, known.

For example, from (Eq 2.67-68), we can see that we can find the solution $\omega$ by knowing one of the $u$ boundaries or the $v$ boundaries, such as $u(0, y)$ and $v(x, 0)$. One may obtain preference to have a particular pair of boundary conditions that make computations easier as we saw in (Eq 2.77-78), for which we chose to have $u(0, y)$ and $v(0, y)$. Therefore, we will quickly discuss the connection between the pairings. As a guiding example, suppose that we have the differential equation:

$$
\begin{align*}
\omega^{\prime} & =\lambda \omega \\
u(x, 0) & =e^{\lambda x}  \tag{2.82}\\
v(x, 0) & =0
\end{align*}
$$

Let us assume that we desire, or maybe even require, the boundary conditions $u(0, y)$ and $v(0, y)$; let us target $u(0, y)$ first to demonstrate. Using (Eq 2.67.1), we can get:

$$
\begin{align*}
\mathcal{L}\{u\} & =\frac{1}{\xi_{1}}[\mathcal{L}\{u(0, y)\}+\lambda \mathcal{L}\{u\}]  \tag{2.83}\\
\rightarrow \quad \mathcal{L}\{u\} & =\frac{1}{\xi_{1}-\lambda} \cdot \mathcal{L}\{u(0, y)\}
\end{align*}
$$

Using (Eq 2.68.1), we can get:

$$
\begin{align*}
\mathcal{L}\{v\} & =\frac{1}{\xi_{2}}[0+\lambda \mathcal{L}\{u\}]  \tag{2.84}\\
\rightarrow \quad \mathcal{L}\{v\} & =\frac{\lambda}{\xi_{2}} \cdot \mathcal{L}\{u\}
\end{align*}
$$

Using (Eq 2.67.2), we can get:

$$
\begin{array}{rlrl} 
& \mathcal{L}\{u\} & =\frac{1}{\xi_{2}}\left[\frac{1}{\xi_{1}-\lambda}-\lambda \mathcal{L}\{v\}\right] \\
\rightarrow \quad \mathcal{L}\{u\} & =\frac{1}{\xi_{2}}\left[\frac{1}{\xi_{1}-\lambda}-\frac{\lambda^{2}}{\xi_{2}} \mathcal{L}\{u\}\right]  \tag{2.85}\\
\rightarrow \quad \mathcal{L}\{u\} & =\frac{\xi_{2}}{\xi_{2}^{2}+\lambda^{2}} \cdot \frac{1}{\xi_{1}-\lambda} .
\end{array}
$$

Equating (Eq 2.83) and (Eq 2.85), we obtain:

$$
\begin{align*}
\mathcal{L}\{u(0, y)\} & =\frac{\xi_{2}}{\xi_{2}^{2}+\lambda^{2}}  \tag{2.86}\\
\rightarrow \quad u(0, y) & =\cos (\lambda y)
\end{align*}
$$

Similiarly, one can obtain that $v(0, y)=\sin (\lambda y)$, which complete the set of boundary conditions for the exact solution of (Eq 2.82) that is $\omega=e^{\lambda z}$. Therefore, using (Eq 2.67) and (Eq 2.68) as basic equations, one can obtain all 4 boundary conditions given two, without having to actually know or find what $u$ and $v$ are equal to.

We can find that if we are given the boundary conditions (two of them at least), then we can find the corresponding initial condition for which we can use the methods described in sections 2.2 and 2.3. For example, suppose that for some $\omega^{\prime}=f(z, \omega)$, we have:

$$
\begin{gathered}
u(x, 3)=x^{2}-\sin (x) \\
v(1, y)=\cos (y)-4 y
\end{gathered}
$$

Setting $x=1$ and $y=3$ can give us the initial condition:

$$
\begin{aligned}
z_{0}=1+3 i & \rightarrow \quad u_{0}=1-\sin (1) \\
z_{0}=1+3 i & \rightarrow \quad v_{0}=\cos (3)-12 \\
\therefore \omega(1+3 i) & =(1-\sin (1))+i(\cos (3)-12) .
\end{aligned}
$$

Suppose we are given an initial condition $\left(z_{0}, \omega_{0}\right)$ and desire the boundary conditions. This problem has already been addressed numerically, in sections 2.2 and 2.3. One can use the LR-method to obtain $u\left(x, y_{0}\right)$ and $v\left(x, y_{0}\right)$ and the BT-method to obtain $u\left(x_{0}, y\right)$ and $v\left(x_{0}, y\right)$ from the initial point $\left(x_{0}, y_{0}\right)$. Yes, this then forces your solution to be numerical since the Laplace transform cannot analytically be used. One can investigate the numerical Laplace transform, but we will leave this for the future work on this idea.

For sake of completion, it is necessary (depending on the audience) to mention the case when the boundary conditions are not prescribed on the axes. For example, the boundary value problem:

$$
\begin{equation*}
\omega^{\prime}=f(z, \omega), u\left(x_{0}, y\right)=\varphi(y), v\left(x, y_{0}\right)=\psi(x) \tag{2.87}
\end{equation*}
$$

can be transformed via $y^{*}:=y-y_{0}$ and $x^{*}:=x-x_{0}$, thus giving a parallel boundary value problem to solve:

$$
\begin{equation*}
\eta^{\prime}=f\left(z^{*}, \eta\right), \mu\left(0, y^{*}\right)=\varphi\left(y+y_{0}\right), \nu(x, 0)=\psi\left(x^{*}+x_{0}\right) \tag{2.88}
\end{equation*}
$$

### 2.6 Existence and Uniqueness Theorem for Complex IVP's

For most of the problems discussed in this chapter, we have been using differential equations with solutions that we already know something about to verify our solutions. Our method converged to the exact solutions and if we were to generate a new arbitrary differential equation with an arbitrary initial condition, then our method will converge to some "solution". The questions that one must answer at that time would be:

- Does the differential equation actually have a solution? If it doesn't, any numerical solution obtained is extraneous.
- If the solution, $\omega_{1}$, exists and the numerical solution converged to $\omega_{1}$, does there exist a solution $\omega_{2}$ that satisfies the IVP that our numerical method does not consider?

We discuss both of these questions and the answers, and also provide a demonstration of it in this section.

Before formally introducing and proving the theorem, let us describe some background terminology. The complex function $\omega$ maps the set $\mathbb{C}$ to $\mathbb{C}$, which we write as:

$$
\omega: \mathbb{C} \rightarrow \mathbb{C}
$$

One may be familiar with an "open ball" $\mathcal{B}_{\epsilon}\left(z_{0}\right)$ to be:

$$
\mathcal{B}_{\epsilon}\left(z_{0}\right):=\left\{z:\left|z-z_{0}\right|<\epsilon\right\}
$$

Here, $\mathcal{B}_{\epsilon}\left(z_{0}\right) \subset \mathbb{C}$. Similarly, one can define the open ball $\mathcal{B}_{\epsilon}\left(z_{0}, \omega_{0}\right)$ to be:

$$
\begin{equation*}
\mathcal{B}_{\epsilon}\left(z_{0}, \omega_{0}\right):=\left\{z, \omega:\left|z-z_{0}\right|<\epsilon,\left|\omega-\omega_{0}\right|<\epsilon\right\} . \tag{2.89}
\end{equation*}
$$

One can see that $\mathcal{B}_{\epsilon}\left(z_{0}, \omega_{0}\right) \subset \mathbb{C}^{2}$. With that being said, the decision to use $z$ and $\omega$ as my variables is by no chance a coincidence. Recall that for the differential equation:

$$
\left.\omega^{\prime}(z)=f(z, \omega(z)), \omega\left(z_{0}\right)=\omega_{0}\right)
$$

the function $f$ satisfies the property:

$$
\begin{equation*}
f: \mathcal{B}_{\epsilon}\left(z_{0}, \omega_{0}\right) \rightarrow \mathbb{C} \tag{2.90}
\end{equation*}
$$

Assume that for some $\kappa \in \mathbb{R}^{+}$that the following property holds:

$$
\begin{array}{r}
\left|f\left(z, \omega_{1}\right)-f\left(z, \omega_{2}\right)\right| \leq \kappa\left|\omega_{1}-\omega_{2}\right|,  \tag{2.91}\\
\forall\left(z, \omega_{1}\right),\left(z, \omega_{2}\right) \in \mathcal{B}_{\epsilon}\left(z_{0}, \omega_{0}\right) .
\end{array}
$$

If (Eq 2.91) satisfies for some $f$ as described in (Eq 2.90), then we call $f$ "Lipshitz continuous." In this section, it will be desired that $f$ is Lipshitz continuous with respect to $\omega$, where $\omega$ is uniformly continuous with respect to $z$. This property will come as a direct result due to the analytic properties of $\omega$, so we take (Eq 2.91) to imply such continuity for this section. We will be interested in the sequence of functions:

$$
\begin{equation*}
\left\{\omega_{k}\right\}_{k=0}^{n} \subset \mathbb{C} \tag{2.92}
\end{equation*}
$$

If this sequence of functions satisfies (Eq 2.91) with the Lipshitz constant $\kappa$ satisfying $\kappa<1$, then we will refer to the sequence ( Eq 2.92 ) as a "contractive sequence" under $f$. By Banach's fixed point theorem (see section 1.2 .1 for a basic real-variable version), the sequence $\left\{\omega_{k}\right\}_{k=0}^{n}$ converges to a unique 'point' in the same metric space; i.e.:

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left(\omega_{k}\right):=\omega \in \mathbb{C} . \tag{2.93}
\end{equation*}
$$

With those points being stated, we now state and prove the existence and uniqueness theorem.

## Theorem 2.6.1. (Existence and Uniqueness Theorem for Complex IVPs)

Let $\mathcal{B}\left(z_{0}, \omega_{0}\right)$ be the open ball located in $\mathbb{C}^{2}$ and suppose $f: \mathcal{B}\left(z_{0}, \omega_{0}\right) \rightarrow \mathbb{C}$ is Lipshitz continuous with respect to $z$, independent of $\omega$, with Lipshitz constant $\kappa$. Then there exists an open ball $\mathcal{O}\left(z_{0}\right) \subset \mathbb{C}$ such that the complex differential equation:

$$
\omega^{\prime}(z)=f(z, \omega(z)), \quad \omega\left(z_{0}\right)=\omega_{0}
$$

has a unique solution in $\mathcal{O}\left(z_{0}\right)$.

Proof. Consider an $r \in \mathbb{R}^{+}$and define the closed ball of size $r$ centered at $z_{0}$ to be:

$$
\mathcal{O}_{r}\left(z_{0}\right):=\left\{z:\left|z-z_{0}\right| \leq r\right\} .
$$

Since $\mathcal{B}\left(z_{0}, \omega_{0}\right)$ is open, we can find numbers $a, b \in \mathbb{R}^{+}$such that the ball:

$$
B\left(z_{0}, \omega_{0}\right):=\left\{z, \omega:\left|z-z_{0}\right| \leq a,\left|\omega-\omega_{0}\right| \leq b \mid\right\}
$$

satisfies the property:

$$
B\left(z_{0}, \omega_{0}\right) \subset \mathcal{B}\left(z_{0}, \omega_{0}\right)
$$

For each real number $r$ with $r \leq a$, define $X_{r}$ to be the subspace of $C\left(\mathcal{O}_{r}\right)$ that consists of all continuous functions $\omega: \mathcal{O}_{r} \rightarrow \mathbb{C}$ such that:

$$
\left|\omega(z)-\omega_{0}\right| \leq b \forall z \in \mathcal{O}_{r} .
$$

For such functions $\omega \in X_{r}$, define the function (which can be viewed as a transformation)
$T(\omega) \in C\left(\mathcal{O}_{r}\right)$ by:

$$
T(\omega(z)):=\omega_{0}+\int_{\gamma} f(z, \omega(z)) d z, \quad \forall z \in \mathcal{O}_{r}
$$

Since $f$ is assumed to be (Lipshitz) continuous in $\mathcal{B}\left(z_{0}, \omega_{0}\right)$, the path $\gamma(z)$ that connects the point $z_{0}$ to the variable point $z$ and the value of the complex integral are independent of one another, hence we can take $\int_{\gamma}$ to simply be a line integral. Hence the transformation $T$ can be written as:

$$
\begin{equation*}
T(\omega(z))=\omega_{0}+\int_{z_{0}}^{z} f(z, \omega(z)) d z \tag{2.94}
\end{equation*}
$$

As a side note, (Eq 2.94) is not a "magical" equation, it is just the associated implicit integral equation that is equivalent to that of the differential equation of interest (aside from the transformation $T$ ).

Since $B\left(z_{0}, \omega_{0}\right) \subset \mathbb{C}^{2}$ is a closed compact set and also since $f$ is a continuous function, then there exists a $K \in \mathbb{R}^{+}$such that:

$$
|f(z, \omega(z))| \leq K, \quad \forall(z, \omega) \in B\left(z_{0}, \omega_{0}\right)
$$

Therefore, for $\omega \in X_{r}$ and $z \in \mathcal{O}_{r}$, we have that:

$$
\begin{align*}
\left|T(\omega(z))-\omega_{0}\right| & =\left|\int_{z_{0}}^{z} f(z, \omega(z)) d z\right| \\
& \leq \int_{z_{0}}^{z}|f(z, \omega(z))| d z  \tag{2.95}\\
& \leq K \cdot \int_{z_{0}}^{z} d z \\
& \leq K \cdot r
\end{align*}
$$

which implies that $T\left(X_{r}\right) \subset X_{r}$ if $K \cdot r \leq b$. In the assumption of the theorem $g$ has Lipshitz
constant $\kappa$, therefore for $\omega_{1}, \omega_{2} \in X_{r}$ and $z \in \mathcal{O}_{r}$, we have that:

$$
\begin{equation*}
\left|f\left(z, \omega_{1}(z)\right)-f\left(z, \omega_{2}(z)\right)\right| \leq \kappa \cdot\left\|\omega_{1}-\omega_{2}\right\| . \tag{2.96}
\end{equation*}
$$

By the properties of line integrals, we have:

$$
\begin{align*}
\left|T\left(\omega_{1}(z)\right)-T\left(\omega_{2}(z)\right)\right| & =\left|\int_{z_{0}}^{z}\left[f\left(z, \omega_{1}(z)\right)-f\left(z, \omega_{2}(z)\right)\right] d z\right|  \tag{2.97}\\
& \leq r \cdot \kappa \cdot\left\|\omega_{1}-\omega_{2}\right\|
\end{align*}
$$

Given (Eq 2.97) and the implications of (Eq 2.95) gives us that $T: X_{r} \rightarrow X_{r}$ is a contraction, provided that $r K \leq b$ and $r \kappa<1$. Define:

$$
\begin{equation*}
r:=\min \{b / K, 1 /(2 \kappa)\} \tag{2.98}
\end{equation*}
$$

Thus by Banachs fixed point theorem, $T: X_{r} \rightarrow X_{r}$ has a unique fixed point.

To illustrate this idea, consider the following differential equation:

$$
\begin{equation*}
\omega^{\prime}(z)=-i \omega, \omega(0)=i \tag{2.99}
\end{equation*}
$$

It is obvious to see that $f$ is continuous on all of $\mathbb{C}$, so the choose of contour $\gamma$ can be chosen to just be the line from 0 to $z$, hence there exists a unique solution to this differential equation. We now define the sequence $\left\{\varphi_{k}\right\}_{k=0}^{n}$ to be:

$$
\begin{align*}
\varphi_{0} & :=\omega_{0} \\
\rightarrow \varphi_{0} & =i \\
\varphi_{k+1} & :=\omega_{0}+\int_{z_{0}}^{z} f\left(t, \varphi_{k}\right) d t  \tag{2.100}\\
\rightarrow \varphi_{k+1} & =i+\int_{0}^{z}\left(-i \cdot \varphi_{k}\right) d t .
\end{align*}
$$

This then gives us:

$$
\begin{aligned}
\varphi_{1} & =i-i \cdot \int_{0}^{z}(i) d z \\
& =i+z
\end{aligned}
$$

The second iteration will give us:

$$
\begin{aligned}
\varphi_{2} & =i-i \cdot \int_{0}^{z}(i+z) d z \\
& =i+z+\frac{i}{2} z^{2} .
\end{aligned}
$$

The third iteration will give us:

$$
\begin{aligned}
\varphi_{3} & =i-i \cdot \int_{0}^{z}\left(i+z+\frac{i}{2} z^{2}\right) d z \\
& =i+z-\frac{i}{2!} z^{2}+\frac{1}{3!} z^{3}
\end{aligned}
$$

and so on. As with power-series methods for solving differential equations, the difficulty usually arises with obtaining a closed form for $\varphi_{k}$ to take the limit of. Keep in mind, that just because the solution exists, the closed form of $\varphi_{k}$ may not necessarily exist (currently). For this case, one can easily verify (add 1 , subtract 1 ) by induction that:

$$
\begin{equation*}
\varphi_{k}=\sum_{l=0}^{k}\left(\frac{z^{2 l}}{(2 l)!}\right)-i \cdot \sum_{l=0}^{k}\left(\frac{z^{2 l-1}}{(2 l-1)!}\right) . \tag{2.101}
\end{equation*}
$$

Therefore the solution of the differential equation is given by:

$$
\begin{align*}
\lim _{k \rightarrow \infty} \varphi_{k}:=\omega & =\cos (z)-i \sin (z)  \tag{2.102}\\
& \equiv e^{-i z}
\end{align*}
$$

This method is not practical by hand, but may give rise to another way to solve complex differential equations once a numerical complex integration method has been developed (this is already a work in progress).

## Chapter 3 DIFFERENTIAL EQUATIONS OF HIGHER ORDER

In Chapter 2, we discussed the first-order complex-variable differential equation. We discussed both an analytical method to solve it via Laplace transforms, and also discussed a numerical method that solves it in two different directions. In this chapter, we discuss the extensions of those concepts to higher orders, modifications that can be performed on them and the consequences thereof.

### 3.1 The Higher-Order Complex Differential Operator

One can view a derivative of a function as a linear operator or mapping. Through this perspective, we can think of higher-order derivatives as compositions of that operator. For example, the function $f(z)=z^{2}$ gets sent to the function $f^{\prime}(z)=2 z$ under the action of the operation $\frac{d}{d z}$, for which is defined as the limit of the difference quotient in the traditional sense. From here on, we will simply denote the operator $\frac{d}{d z}$ as:

$$
\begin{equation*}
\mathcal{D}:=\frac{d}{d z} \tag{3.1}
\end{equation*}
$$

Here, we assume that the domain of $\mathcal{D}$ is the set, $\Omega$, of differential functions $\omega \in \Omega$ of complex variable $z=x+i y$, where $x, y \in \mathbb{R}$.

In Chapter 1 and Chapter 2, we discussed and took advantage of the fact that the operator $\mathcal{D}$ can be written in terms of the real- and imaginary-parts of the functions which are then function of the two real variables $x, y$. That is, for some $\varphi \in \mathbb{R}^{2}$ :

$$
\begin{equation*}
\mathcal{D}(\varphi)=\frac{1}{2} \cdot\left(\varphi_{x}-i \varphi_{y}\right) \tag{3.2}
\end{equation*}
$$

The next natural question to ask is 'what would a similar representation for the second derivative be?'. Composing the operator $\mathcal{D}$ with itself, we can obtain:

$$
\begin{align*}
\mathcal{D}^{2}(\varphi): & =\mathcal{D}(\mathcal{D}(\varphi)) \\
& =\frac{1}{2} \cdot \mathcal{D}\left(\varphi_{x}-i \varphi_{y}\right)  \tag{3.3}\\
& =\frac{1}{2^{2}} \cdot\left(\varphi_{x x}-2 i \varphi_{x y}-\varphi_{y y}\right)
\end{align*}
$$

Keep in mind, that $\varphi$ will take place of the functions $u(x, y)$ and $v(x, y)$ of our function $\omega$. For $\varphi_{x y}$ and $\varphi_{y x}$ to be equal, the function $\varphi \in L^{2}(x, y)$. This is guaranteed since $\omega$ is analytic. Continuing in the same fashion, we have:

$$
\begin{align*}
\mathcal{D}^{3}(\varphi) & =\frac{1}{2^{2}} \cdot\left[\mathcal{D}\left(\varphi_{x x}\right)-2 i \mathcal{D}\left(\varphi_{x y}\right)-\mathcal{D}\left(\varphi_{y y}\right)\right]  \tag{3.4}\\
& =\frac{1}{2^{3}} \cdot\left(\varphi_{x x x}-3 i \varphi_{x x y}-3 \varphi_{x y y}+i \varphi_{y y y}\right)
\end{align*}
$$

For notation purposes, let us denote:

$$
\begin{equation*}
\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{\partial^{\beta}}{\partial y^{\beta}}(\varphi(x, y))\right):=\varphi^{(\alpha, \beta)}(x, y) \tag{3.5}
\end{equation*}
$$

From (Eq 3.2-4), one can show by mathematical induction that the following identity holds for $k \in \mathbb{N}$ :

$$
\begin{equation*}
D^{k}(\varphi)=\left(\frac{1}{2}\right)^{k} \cdot \sum_{l=0}^{k}\binom{k}{l}(-i)^{l} \cdot \varphi^{(k-l, l)} \tag{3.6}
\end{equation*}
$$

One may recall the generalized Cauchy integral formula:

$$
\begin{equation*}
\omega^{(k)}(z)=\frac{k!}{2 \pi i} \cdot \int_{\gamma} \frac{\omega(\zeta)}{(\zeta-z)^{k+1}} d \zeta, \tag{3.7}
\end{equation*}
$$

which holds true if $\omega$ is analytic on and inside of the closed contour $\gamma$ for all $z \in \gamma$, and one may question the connection, if any, there is to (Eq 2.6). Rewriting $\omega$ as $u+i v$ and implementing (Eq 3.6), we have the following identity:

$$
\begin{equation*}
\omega^{(k)}(z)=\left(\frac{1}{2}\right)^{k} \cdot \sum_{l=0}^{k}\left[\binom{k}{l}(-i)^{l}\left(u^{(k-l, l)}+i v^{(k-l, l)}\right)\right] . \tag{3.8}
\end{equation*}
$$

Equating (Eq 3.7) and (Eq 3.8), after rearranging we obtain:

$$
\begin{equation*}
\int_{\gamma} \frac{\omega(\zeta)}{(\zeta-z)^{k+1}} d \zeta=\frac{\pi i}{2^{k-1}} \cdot \sum_{l=0}^{k} \frac{(-i)^{l}}{l!(k-l)!}\left(u^{(k-l, l)}+i v^{(k-l, l)}\right), k \in \mathbb{N} . \tag{3.9}
\end{equation*}
$$

This section provides two useful things which we will discuss in the upcoming sections:

- A real two-variable partial differential equation that is associated to a complex-variable equation (Eq 3.6). We will discuss this equation from the numerical perspective and also the theoretical perspective via Laplace transforms.
- A new method to calculate contour integrals (Eq 3.9).


### 3.2 Numerical Solutions to Higher Order Equations

In the last section, we derived a general form for the $k^{t h}$-order complex-variable differential operator:

$$
\omega^{(k)}(z)=\left(\frac{1}{2}\right)^{k} \cdot \sum_{l=0}^{k}\left[\binom{k}{l}(-i)^{l}\left(u^{(k-l, l)}+i v^{(k-l, l)}\right)\right] .
$$

In this section, we will make use of this method as part of a numerical scheme for solving higher order differential equations. As a demonstration, we will first consider the general second-order linear inhomogeneous differential:

$$
\begin{equation*}
\omega^{\prime \prime}+\alpha(z) \cdot \omega^{\prime}+\beta(z) \cdot \omega=\gamma(z) \tag{3.10}
\end{equation*}
$$

We will assume that the functions $\alpha, \beta, \gamma \in L^{2}(\mathbb{C})$ are known and they have corresponding real and complex parts $\alpha_{r}, \beta_{r}, \gamma_{r}, \alpha_{c}, \beta_{c}, \gamma_{c} \in L^{2}\left(\mathbb{R}^{2}\right)$, respectively. Recall that from (Eq 3.2) and (Eq 3.3), we have after rearranging:

$$
\begin{align*}
\omega & =u+i v \\
\omega^{\prime} & =\frac{1}{2}\left(u_{x}+v_{y}\right)+\frac{i}{2}\left(v_{x}-u_{y}\right)  \tag{3.11}\\
\omega^{\prime \prime} & =\frac{1}{4}\left(u_{x x}-u_{y y}+2 v_{x y}\right)+\frac{i}{4}\left(v_{x x}-v_{y y}-2 u_{x y}\right) .
\end{align*}
$$

Multiplying $(\operatorname{Eq} 3.11 .2)$ by $\alpha(z)=\alpha_{r}(x, y)+i \alpha_{c}(x, y)$ and $(\operatorname{Eq} 3.11 .1)$ by $\beta(z)=\beta_{r}(x, y)+$ $i \beta_{c}(x, y)$, we have after rearranging ( Eq 3.12 ).

$$
\begin{align*}
\beta(z) \cdot \omega & =\left(\beta_{r} u-\beta_{c} v\right)+i\left(\beta_{c} u+\beta_{r} v\right) \\
\alpha(z) \cdot \omega^{\prime} & =\frac{1}{2}\left[\alpha_{r}\left(u_{x}+v_{y}\right)+\alpha_{c}\left(u_{y}-v_{x}\right)\right]+\frac{i}{2}\left[\alpha_{r}\left(v_{x}-u_{y}\right)+\alpha_{c}\left(u_{x}+v_{y}\right)\right]  \tag{3.12}\\
\omega^{\prime \prime} & =\frac{1}{4}\left(u_{x x}-u_{y y}+2 v_{x y}\right)+\frac{i}{4}\left(v_{x x}-v_{y y}-2 u_{x y}\right) .
\end{align*}
$$

Summing up the real- and imaginary-parts of (Eq 3.12), setting equal to $\gamma(z)=\gamma_{r}(x, y)+i \gamma_{c}(x, y)$, multiplying both sides by 4 and equating real and imaginary parts gives us the system:

$$
\begin{align*}
& \left(u_{x x}-u_{y y}+2 v_{x y}\right)+2 \alpha_{r}\left(u_{x}+v_{y}\right)+2 \alpha_{c}\left(u_{y}-v_{x}\right)+4 \beta_{r} u-4 \beta_{c} v=4 \gamma_{r}  \tag{3.13}\\
& \left(v_{x x}-v_{y y}-2 u_{x y}\right)+2 \alpha_{r}\left(v_{x}-u_{y}\right)+2 \alpha_{c}\left(u_{x}+v_{y}\right)+4 \beta_{c} u+4 \beta_{r} v=4 \gamma_{c} .
\end{align*}
$$

Since $\omega$ is an analytic function, its real and imaginary parts $u$ and $v$ have continuous partial derivatives of all orders. Since this is true, we can interchange the order of partial derivatives for each of them; i.e. $\varphi_{x y}=\varphi_{y x}$. With this in mind, we will choose ( Eq 3.13 .1 ) to be an equation dominantly focused on $u$ and ( Eq 3.13 .2 ) to be an equation dominantly focused on $v$. One can verify from the CR equations that:

$$
v_{x y}=u_{x x} \quad u_{x y}=-v_{x x}
$$

Using these identities together with the CR equations, ( Eq 3.13 ) is then equal to:

$$
\begin{align*}
& \left(3 u_{x x}-u_{y y}\right)+4 \alpha_{r} u_{x}+4 \alpha_{c} u_{y}+4 \beta_{r} u-4 \beta_{c} v=4 \gamma_{r}  \tag{3.14}\\
& \left(3 v_{x x}-v_{y y}\right)+4 \alpha_{r} v_{x}+4 \alpha_{c} v_{y}+4 \beta_{c} u+4 \beta_{r} v=4 \gamma_{c} .
\end{align*}
$$

Consider the initial conditions $\omega\left(z_{0}\right)=u_{0}+i v_{0}$ and $\omega^{\prime}\left(z_{0}\right)=u_{1}+i v_{1}$. Since both equations have both $u$ and $v$ (see last two terms of the left-hand side of ( Eq 3.14 )), the equations are coupled and must be solved simultaneously. For notation purposes, denote the initial conditions and for-ward differences by:

$$
\begin{array}{rl}
u\left(x_{0}, y_{0}\right) \equiv u_{0}:=u^{1,1} & v\left(x_{0}, y_{0}\right) \equiv v_{0}:=v^{1,1} \\
u\left(x_{0}+j h, y_{0}+k h\right):=u^{j+1, k+1} & v\left(x_{0}+j h, y_{0}+k h\right):=v^{j+1, k+1} . \tag{3.16}
\end{array}
$$

Note here that the computational domain $\mathcal{D}$ is consistently and uniformly spaced. For (Eq 3.14) we will use a 2 nd-order approximation for all derivatives present; namely:

$$
\begin{equation*}
u_{x x} \approx \frac{1}{h^{2}}\left(u^{j+1, k}-2 u^{j, k}+u^{j-1, k}\right), \quad u_{x} \approx \frac{1}{2 h}\left(u^{j+1, k}-u^{j-1, k}\right), \tag{3.17}
\end{equation*}
$$

and similarly for $v_{x x}, v_{x}$ and the $y$ derivatives. This then transforms (Eq 3.14) into:

$$
\begin{align*}
& \frac{3}{h^{2}}\left(u^{j+1, k}-2 u^{j, k}+u^{j-1, k}\right)-\frac{1}{h^{2}}\left(u^{j, k+1}-2 u^{j, k}+u^{j, k-1}\right)+\frac{2}{h} \alpha_{r}\left(x_{j}, y_{k}\right)\left(u^{j+1, k}-u^{j-1, k}\right) \\
& \quad+\frac{2}{h} \alpha_{c}\left(x_{j}, y_{k}\right)\left(u^{j, k+1}-u^{j, k-1}\right)+4 \beta_{r}\left(x_{j}, y_{k}\right) u^{j, k}-4 \gamma_{r}\left(x_{j}, y_{k}\right)=4 \beta_{c}\left(x_{j}, y_{j}\right) v^{j, k}  \tag{3.18}\\
& \frac{3}{h^{2}}\left(v^{j+1, k}-2 v^{j, k}+v^{j-1, k}\right)-\frac{1}{h^{2}}\left(v^{j, k+1}-2 v^{j, k}+v^{j, k-1}\right)+\frac{2}{h} \alpha_{r}\left(x_{j}, y_{k}\right)\left(v^{j+1, k}-v^{j-1, k}\right) \\
& \quad+\frac{2}{h} \alpha_{c}\left(x_{j}, y_{k}\right)\left(v^{j, k+1}-v^{j, k-1}\right)+4 \beta_{r}\left(x_{j}, y_{k}\right) v^{j, k}-4 \gamma_{c}\left(x_{j}, y_{k}\right)=-4 \beta_{c}\left(x_{j}, y_{j}\right) u^{j, k} . \tag{3.19}
\end{align*}
$$

As an observation, that aside from $u$ and $v$, the only difference between ( Eq 3.18 ) and (Eq 3.19) is the terms surrounding the equal sign. Rearranging these equations into a more useful form, we have:

$$
\begin{align*}
& u^{j+1, k}\left(3+2 h \alpha_{r}^{j, k}\right)+u^{j, k+1}\left(-1+2 h \alpha_{c}^{j, k}\right) \\
&= v^{j, k}\left(4 h^{2} \beta_{c}^{j, k}\right)+u^{j, k}\left(4-4 h^{2} \beta_{r}^{j, k}\right)  \tag{3.20}\\
&+u^{j-1, k}\left(-3+2 h \alpha_{r}^{j, k}\right)+u^{j, k-1}\left(1+2 h \alpha_{c}^{j, k}\right) \\
&+4 h^{2} \gamma_{r}^{j, k} . \\
& v^{j+1, k}\left(3+2 h \alpha_{r}^{j, k}\right)+v^{j, k+1}\left(-1+2 h \alpha_{c}^{j, k}\right) \\
&= u^{j, k}\left(-4 h^{2} \beta_{c}^{j, k}\right)+v^{j, k}\left(4-4 h^{2} \beta_{r}^{j, k}\right)  \tag{3.21}\\
&+v^{j-1, k}\left(-3+2 h \alpha_{r}^{j, k}\right)+v^{j, k-1}\left(1+2 h \alpha_{c}^{j, k}\right) \\
&+4 h^{2} \gamma_{c}^{j, k} .
\end{align*}
$$

For sake of discussion, denote the right-hand side of (Eq 3.20) as $\Phi_{u}^{j, k}$ and the right-hand side of (Eq 3.21) as $\Phi_{v}^{j, k}$, making our finite-difference systems equivalent to that of:

$$
\begin{aligned}
& u^{j+1, k}\left(3+2 h \alpha_{r}^{j, k}\right)+u^{j, k+1}\left(-1+2 h \alpha_{c}^{j, k}\right)=\Phi_{u}^{j, k} \\
& v^{j+1, k}\left(3+2 h \alpha_{r}^{j, k}\right)+v^{j, k+1}\left(-1+2 h \alpha_{c}^{j, k}\right)=\Phi_{v}^{j, k} .
\end{aligned}
$$

To implement this numerical scheme, when $j$ or $k$ is equal to 2 , the knowledge of $u^{0,0}$ and $v^{0,0}$ is needed, which exists before our initial condition. From these "ghost points", we can take two half-steps of size $\frac{h}{2}$ to get to our initial conditions. To implement this recall that from (Eq 3.2):

$$
\omega^{\prime}(z)=\frac{1}{2}\left(u_{x}+v_{y}\right)+\frac{i}{2}\left(v_{x}-u_{y}\right) .
$$

From our second initial condition to the general inhomogeneous differential equation we have:

$$
\begin{equation*}
\omega^{\prime}\left(z_{0}\right)=\frac{1}{2}\left(u_{x}\left(x_{0}, y_{0}\right)+v_{y}\left(x_{0}, y_{0}\right)\right)+\frac{i}{2}\left(v_{x}\left(x_{0}, y_{0}\right)-u_{y}\left(x_{0}, y_{0}\right)\right) \equiv u_{1}+i v_{1} . \tag{3.22}
\end{equation*}
$$

Using the Cauchy-Riemann equations to restructure (Eq 3 .22) and equating real and imaginary parts, we have the following:

$$
\begin{array}{ccc}
\frac{u\left(x_{0}, y_{0}\right)-u\left(x_{0}-h, y_{0}\right)}{h}=u_{1} & \rightarrow & u^{0,1}=u^{1,1}-h u_{1} \\
\frac{-u\left(x_{0}, y_{0}\right)+u\left(x_{0}, y_{0}-h\right)}{h}=v_{1} & \rightarrow \quad u^{1,0}=u^{1,1}+h v_{1} \\
\frac{v\left(x_{0}, y_{0}\right)-v\left(x_{0}-h, y_{0}\right)}{h}=v_{1} & \rightarrow & v^{0,1}=v^{1,1}-h v_{1}  \tag{3.23}\\
\frac{v\left(x_{0}, y_{0}\right)-v\left(x_{0}, y_{0}-h\right)}{h}=u_{1} & \rightarrow & v^{1,0}=v^{1,1}-h u_{1} .
\end{array}
$$

To demonstrate this method, let us consider the constant-coefficient second-order non-homogeneous differential equation:

$$
\begin{equation*}
\omega^{\prime \prime}+7 \omega^{\prime}+10 \omega=i, \quad \omega(0)=\frac{i}{10}, \quad \omega^{\prime}(0)=1+2 i . \tag{3.24}
\end{equation*}
$$

Applying (Eq 3.14) onto this particular equation gives us the 2 -variable PDE system:

$$
\begin{align*}
3 u_{x x}-u_{y y}+28 u_{x}+40 u & =0  \tag{3.25}\\
3 v_{x x}-v_{y y}+28 v_{x}+40 v & =4 .
\end{align*}
$$

Note here that these equations are uncoupled and can be solved for $u$ and $v$ independently, hence any error accumulation will also be dependent solely only the initial steps and its consequences. Applying (Eq 3.20) and (Eq 3.21) to obtain our numerical scheme, we obtain:

$$
\begin{align*}
& u^{j+1, k}(3+14 h)-u^{j, k+1}=u^{j, k}\left(4-40 h^{2}\right)+u^{j-1, k}(-3+14 h)+u^{j, k-1}  \tag{3.26}\\
& v^{j+1, k}(3+14 h)-v^{j, k+1}=v^{j, k}\left(4-40 h^{2}\right)+v^{j-1, k}(-3+14 h)+v^{j, k-1}+4 h^{2} .
\end{align*}
$$

Our initial condition $\omega(0)$ gives us:

$$
\begin{equation*}
u_{0} \equiv u^{1,1}=0 \quad v_{0} \equiv v^{1,1}=0.2 \tag{3.27}
\end{equation*}
$$

and initial condition $\omega^{\prime}(0)$ (i.e. $\left.u_{1}=1, v_{1}=2\right)$ together with ( $\operatorname{Eq} 3.23$ ) and ( Eq 3.27 ) gives:

$$
\begin{array}{cc}
u^{0,1}=-h, & u^{1,0}=2 h \\
v^{0,1}=0.2-2 h, & v^{1,0}=2-h . \tag{3.28}
\end{array}
$$

This gives us all of the tools to implement our LR and BT methods as described in the previous chapter to obtain a numerical solution for the differential equation. To compare our numerical solution, we will compare to the easily obtainable analytical soltuion:

$$
\begin{equation*}
\omega(z)=c_{1} e^{-5 z}+c_{2} e^{-2 z}+\frac{i}{10}, \quad \text { where } c=\binom{-1 / 3-2 / 3 i}{1 / 3+2 / 3 i} . \tag{3.29}
\end{equation*}
$$

To graphically illustrate the LT and BT methods with single ghost cells, consider the following diagram. Depending on the general differential equation, different schemes or approaches. Take


Figure 3.1: Solution Strategies for $1 \mathbb{C}-\mathrm{D}$ w/ Single Ghost Cells
for example, the LR method initialization phase. The first point for which we will desire will be $u^{2,1}$, since $u^{1,1}$ is given as an initial condition and $u^{0,1}$ is a ghost cell that allows for the second order scheme to be set up. Letting $j=k=1$ in (Eq 3.26), we obtain an equation containing $u^{2,1}$ (what we want), on the RHS we have $u^{1,1}, u^{0,1}$ and $u^{1,0}$, but on the LHS, we have the term $u^{1,2}$. By the defined LR method, this point in the solution space is not calculated until the solution phase (post initialization). Note that this "issue" will occur any time a "Laplacian"-like equation is being solved with second-order in both spatial dimensions.

One could approach this in one of many ways, the two obvious approaches are:

- Create a boundary of ghost cells on both sides of the computational domain and use them to approximate the boundary conditions.
- Approximate the inner solution with a first-order scheme with spatial steps $\frac{h}{2}$ to approximate the boundary.

We will take the second approach. With this approach, one needs to consider a couple things since the part of the solution space (besides the boundary) have been approximation in the initialization of LR/BT:

- Do you keep the first column (LR)/first row (BT) that was used to calculate the boundary?
- Should the ghost-cells be stored during the initialization phase for the solution phase of both LR and BT?

The answer (in my approach) to both of these questions is 'yes'. Recalculating the first column/row from the boundary is equivalent as using the first column/row to calculate the boundary. For the second question, the vertical ghost cells are used in initialization of LR and used in the solution phase of BT, whereas the horizontal ghost cells are used in the initialization of BT and used in the solution phase of LR. To explain the initialization phase (of LR in particular), consider the following diagrams. For the solution phase of LR (and BT), as one proceeds across the row (column) by say $n$ steps, one needs to know the $n-1$ steps of the next row above (column to the right). In short, a bit more complex than the first-order methods described in Chapter 2, but that is the trade off needed for higher order accuracy. See the diagrams on the following page for details.

As the reader may already be thinking, this approach is not practical, especially since it is not for the general equation of interest. Yes; the purpose of this section is to just pose that "it can be done" via this approach. But for those who are familiar with the sequence of differential topics, the next section is a trivial follow up approach to higher-order equations.


Figure 3.2: Second-Order Strategy for LR(BT) Initialization Phase


Figure 3.3: Second-Order Strategy for LR(BT) Solution Phase

### 3.3 First-Order System Approach for Higher Order Equations

As we have seen previously, there are many issues that make a direct approach to solving higher order differential equations complication. With this in mind, a convenient approach to higher-order equations is by converting them into a system of first-order equations. Let us take (Eq 3.24) as a guiding example of this process:

$$
\omega^{\prime \prime}+7 \omega^{\prime}+10 \omega=i, \quad \omega(0)=\frac{i}{10}, \quad \omega^{\prime}(0)=1+2 i .
$$

The rule here is that for the highest derivative in the equation, that corresponds to how many new complex variables to introduce and define; in this case, the order is 2 . Define the following complex variables:

$$
\begin{align*}
& \eta_{1}:=\omega  \tag{3.30}\\
& \eta_{2}:=\omega^{\prime}
\end{align*}
$$

Note at this point, that the only variable that will 'matter' is the variable $\eta_{1}$, where $\eta_{2}$ (and onwards for higher orders) will only be used to obtain $\eta_{1}$. From (Eq 3.30) one can easily see that $\eta_{1}^{\prime}=\eta_{2}$. In general $\eta_{k}^{\prime}=\eta_{k+1}$. From this identity and (Eq 3.24) we obtain an equivalent first-order system for our differential equation:

$$
\begin{align*}
& \eta_{1}^{\prime}=\eta_{2}  \tag{3.31}\\
& \eta_{2}^{\prime}=-10 \eta_{1}-7 \eta_{2}+i .
\end{align*}
$$

It is sometimes convenient to write this in matrix form as:

$$
\binom{\eta_{1}}{\eta_{2}}^{\prime}=\left(\begin{array}{cc}
0 & 1  \tag{3.32}\\
-10 & -7
\end{array}\right)\binom{\eta_{1}}{\eta_{2}}+\binom{0}{i} .
$$

For the general higher-order system:

$$
\begin{equation*}
\omega^{(n)}+\alpha_{n-1} \omega^{(n-1)}+\ldots+\alpha_{0} \omega=\beta \tag{3.33}
\end{equation*}
$$

one has the corresponding first-order system:

$$
\left(\begin{array}{c}
\eta_{1}  \tag{3.34}\\
\eta_{2} \\
\eta_{3} \vdots \eta_{n-1} \\
\eta_{n}
\end{array}\right)^{\prime}=\left(\begin{array}{cccc}
0 & 1 & 0 \ldots 0 & 0 \\
0 & 0 & 1 \ldots 0 & 0 \\
0 \vdots 0 & 0 \vdots 0 & 0 \vdots 0 & 0 \vdots 1 \\
-\alpha_{0} & -\alpha_{1} & -\alpha_{2} \ldots-\alpha_{n-2} & -\alpha_{n-1}
\end{array}\right)\left(\begin{array}{c}
\eta_{1} \\
\eta_{2} \\
\eta_{3} \vdots \eta_{n-1} \\
\eta_{n}
\end{array}\right)+\left(\begin{array}{c}
0 \\
0 \\
0 \vdots 0 \\
\beta
\end{array}\right) .
$$

Here, $\alpha_{k}$ and $\beta$ can be functions $\mathbb{C} \rightarrow \mathbb{C}$, not necessarily constants (see next example). From here, we will temporarily refer to the real and imaginary parts of $\eta_{k}(z)$ by $a_{k}(x, y)$ and $b_{k}(x, y)$, respectively. One can then obtain, from the Wirtinger representation of $\frac{\partial}{\partial z}$, the following two identities:

$$
\begin{align*}
& \left(\frac{\partial}{\partial x}-i \frac{\partial}{\partial y}\right)\left(a_{k}+i b_{k}\right)=-\left(a_{k+1}+i b_{k+1}\right), k=1,2, \ldots, n-1  \tag{3.35}\\
& \left(\frac{\partial}{\partial x}-i \frac{\partial}{\partial y}\right)\left(a_{n}+i b_{n}\right)=\beta-\sum_{l=0}^{n-1} \alpha_{l} \cdot\left(a_{l}+i b_{l}\right) \tag{3.36}
\end{align*}
$$

Equating real and imaginary parts of (Eq 3.35) gives us:

$$
\begin{align*}
& \left(a_{k}\right)_{x}+\left(b_{k}\right)_{y}=-\left(a_{k+1}\right)  \tag{3.37}\\
& \left(b_{k}\right)_{x}-\left(a_{k}\right)_{y}=-\left(b_{k+1}\right), \quad k=1,2, \ldots, n-1
\end{align*}
$$

and equating real and imaginary parts of (Eq 3.36), where $\beta:=\beta_{r}+i \beta_{c}$ and $\alpha_{l}:=\left(\alpha_{l}\right)_{r}+i\left(\alpha_{l}\right)_{c}$, gives us:

$$
\begin{align*}
& \left(a_{n}\right)_{x}+\left(b_{n}\right)_{y}=\beta_{r}-\sum_{l=0}^{n-1}\left[\left(\alpha_{l}\right)_{r} a_{l}-\left(\alpha_{l}\right)_{c} b_{l}\right] \\
& \left(b_{n}\right)_{x}-\left(a_{n}\right)_{y}=\beta_{c}-\sum_{l=0}^{n-1}\left[\left(\alpha_{l}\right)_{r} b_{l}-\left(\alpha_{l}\right)_{c} a_{l}\right] . \tag{3.38}
\end{align*}
$$

Remember, the goal of converting this to a system is to solve for the functions $a_{1} \equiv u$ and $b_{1} \equiv v$, which builds our $\eta_{1} \equiv \omega$.For our problem, (Eq 3.32), applying the Cauchy-Riemann equations on the real and imaginary parts on $\eta_{1}, \eta_{2}$ following the Wirtinger decomposition, and equating real
and imaginary parts, we obtain the following systems; from ( $\operatorname{Eq} 3.32 .1$ ), we have:

$$
\begin{align*}
\left(a_{1}\right)_{x} & =\frac{1}{2}\left(a_{2}\right) & \left(a_{1}\right)_{y} & =-\frac{1}{2}\left(b_{2}\right)  \tag{3.39}\\
\left(b_{1}\right)_{x} & =\frac{1}{2}\left(b_{2}\right) & \left(b_{1}\right)_{y} & =\frac{1}{2}\left(a_{2}\right)
\end{align*}
$$

and from (Eq 3.32.2) we have:

$$
\begin{align*}
\left(a_{2}\right)_{x}=\frac{1}{2} \cdot\left[-10\left(a_{1}\right)-7\left(a_{2}\right)\right] & \left(a_{2}\right)_{y}=-\frac{1}{2} \cdot\left[-10\left(b_{1}\right)-7\left(b_{2}\right)+1\right]  \tag{3.40}\\
\left(b_{2}\right)_{x}=\frac{1}{2} \cdot\left[-10\left(b_{1}\right)-7\left(b_{2}\right)+1\right] & \left(b_{2}\right)_{y}=\frac{1}{2} \cdot\left[-10\left(a_{1}\right)-7\left(a_{2}\right)\right] .
\end{align*}
$$

For brevity, I won't mention the generalizations from here on since they are trivial to derive. From here, recall that $\omega(z)=\eta_{1}(z)$ and $\omega^{\prime}(z)=\eta_{2}(z)$, therefore $\omega(0)=\eta_{1}(0)=\frac{i}{10}$ and $\omega^{\prime}(0)=\eta_{2}(0)=$ $1+2 i$. This implies that:

$$
\begin{array}{ll}
a_{1}(0,0):=a_{1}^{1,1}=0 & b_{1}(0,0):=b_{1}^{1,1}=0.2 \\
a_{2}(0,0):=a_{2}^{1,1}=1 & b_{2}(0,0):=b_{2}^{1,1}=2 \tag{3.42}
\end{array}
$$

Applying a forward-difference scheme on (Eq 3.39) and (Eq 3.40) gives us:

$$
\begin{align*}
a_{1}^{j+1, k} & =a_{1}^{j, k}+\frac{h}{2} a_{2}^{j, k} & a_{1}^{j, k+1}=a_{1}^{j, k}-\frac{h}{2} b_{2}^{j, k}  \tag{3.43}\\
b_{1}^{j+1, k} & =b_{1}^{j, k}+\frac{h}{2} b_{2}^{j, k} & b_{1}^{j, k+1}=b_{1}^{j, k}+\frac{h}{2} a_{2}^{j, k},
\end{align*}
$$

and

$$
\begin{align*}
a_{2}^{j+1, k}=a_{2}^{j, k}+\frac{h}{2} \cdot\left[-10 a_{1}^{j, k}-7 a_{2}^{j, k}\right] & a_{2}^{j, k+1}=a_{2}^{j, k}-\frac{h}{2} \cdot\left[-10 b_{1}^{j, k}-7 b_{2}^{j, k}+1\right]  \tag{3.44}\\
b_{2}^{j+1, k}=b_{2}^{j, k}+\frac{h}{2} \cdot\left[-10 b_{1}^{j, k}-7 b_{2}^{j, k}+1\right] & b_{2}^{j, k+1}=b_{2}^{j, k}+\frac{h}{2} \cdot\left[-10 a_{1}^{j, k}-7 a_{2}^{j, k}\right] .
\end{align*}
$$

From here, the solution strategy is trivial. For the LR method:

- Initialization Phase: Use (Eq 3.43-4),
- Solution Phase: Use (Eq 3.43-4),
and for the BT method:
- Initialization Phase: Use (Eq 3.43-4),
- Solution Phase: Use (Eq 3.43-4).

As already mentioned in the previous section, we can compare the solutions $a_{1}=\operatorname{Re}\left(\eta_{1}\right)=u, b_{1}=$ $\operatorname{Im}\left(\eta_{1}\right)=v$ to the real and imaginary parts of the analytical solution (Eq 3.29):

$$
\begin{align*}
& u(x, y)=-\frac{1}{3} e^{-5 x} \cos (5 y)-\frac{2}{3} e^{-5 x} \sin (5 y)+\frac{1}{3} e^{-2 x} \cos (2 y)+\frac{2}{3} e^{-2 x} \sin (2 y)  \tag{3.45}\\
& v(x, y)=\frac{1}{3} e^{-5 x} \sin (5 y)-\frac{2}{3} e^{-5 x} \cos (5 y)-\frac{1}{3} e^{-2 x} \sin (2 y)+\frac{2}{3} e^{-2 x} \cos (2 y)+0.2
\end{align*}
$$

The results for the forward difference approach are shown below.


Figure 3.4: Real/Imaginary Parts for Exact Solution of $\omega^{\prime \prime}=-7 \omega^{\prime}-10 \omega+i$

| Example $6 \mathrm{w} / \mathrm{h}=0.1$ | LR Method | BT Method | Example $6 \mathrm{w} / \mathrm{h}=0.005$ | LR Method | BT Method |
| :---: | :--- | :--- | :---: | :--- | :--- |
| $E_{u}^{L 2}$ | 4.7905 | 4.7905 | $E_{u}^{L 2}$ | 3.8324 | 3.8324 |
| $E_{u}^{R P D}$ | -0.0394 | -0.0394 | $E_{u}^{R D}$ | -0.00437 | -0.00437 |
| $E_{v}^{L 2}$ | 6.3629 | 6.3629 | $E_{v}^{L 2}$ | 5.0102 | 5.0102 |
| $E_{v}^{R P D}$ | 0.1590 | 0.1590 | $E_{v}^{R P D}$ | 0.02798 | 0.02798 |

Figure 3.5: Error Measures for Numerical Solutions w/h $=0.1$ and $h=0.005$

Once confidence is placed in some numerical method and verified with known solutions, one can then apply it to other differential equations either for fun or experimentation. For example, complex De Boer-Ludford equation given by:

$$
\begin{equation*}
\omega^{\prime \prime}-z \omega=2 \omega \cdot|\omega|^{\alpha}, \quad \alpha>0 . \tag{3.46}
\end{equation*}
$$

Doing the $\eta_{1}, \eta_{2}$ transformation of (Eq 3.46) gives us the non-linear system:

$$
\begin{align*}
& \eta_{1}^{\prime}=\eta_{2}  \tag{3.47}\\
& \eta_{2}^{\prime}=z \eta_{1}+2 \eta_{1} \cdot\left|\eta_{1}\right|^{\alpha} .
\end{align*}
$$

Here, for any $\eta_{1} \in \mathbb{C} \times \mathbb{C}$ and $\alpha \in \mathbb{R}^{+},\left|\eta_{1}\right|^{\alpha} \in \mathbb{R}$. It is also important to note that for any second-order differential equation, the $\eta_{1}^{\prime}$ equation can be decomposed into ( Eq 3.39 ). For our

De-Boer-Ludford equation, we have that the $\eta_{2}^{\prime}$ equation decomposition is given by:

$$
\begin{align*}
\left(a_{2}\right)_{x} & =\frac{1}{2}\left(x a_{1}-y b_{1}+2 a_{1}\left|a_{1}+i b_{1}\right|^{\alpha}\right) \\
\left(b_{2}\right)_{x} & =\frac{1}{2}\left(x b_{1}+y a_{1}+2 b_{1}\left|a_{1}+i b_{1}\right|^{\alpha}\right)  \tag{3.48}\\
\left(a_{2}\right)_{y} & =-\frac{1}{2}\left(x b_{1}+y a_{1}+2 b_{1}\left|a_{1}+i b_{1}\right|^{\alpha}\right) \\
\left(b_{2}\right)_{y} & =\frac{1}{2}\left(x a_{1}-y b_{1}+2 a_{1}\left|a_{1}+i b_{1}\right|^{\alpha}\right) .
\end{align*}
$$

One can then apply a forward-difference scheme as was done in (Eq 3.43-44) to obtain a numerical solution.

The Riemann surface for various values of $\alpha$ under the initial condition $\omega(0)=i$ and $\omega^{\prime}(0)=2-5 i$ are shown below. It is to note that the De Boer-Ludford equation is typically treated as a boundary value problem with complex-extended boundary conditions given by:

$$
\begin{array}{r}
\omega(\infty+i \infty)=0  \tag{3.49}\\
\omega(z) \sim-(0.5 z)^{1 / \alpha},
\end{array}
$$

but as our focus here is not BVP's, but rather IVP's, we reserve this type of problem for our future consideration. One can observe from the Riemann surfaces why such boundary conditions may want to be imposed. More about the De Boer-Ludford equation and its application to plasma physics can be found in [24].


Figure 3.6: De Boer-Ludford Riemann Surface, $\alpha=0.1$ and $\alpha=0.8$


Figure 3.7: De Boer-Ludford Riemann Surface, $\alpha=1.2$ and $\alpha=1.8$

### 3.4 Laplace Transform Solutions to Higher-Order Equations

Now that we have discussed numerical solutions to higher-order differential equations, we again return to the discussion of Laplace transforms to help us obtain the exact solution to some equations (namely the ones whose Laplace transform are practically obtainable). Before we proceed to generalizations, let us just recall some of the important results that we derived in section 2.4. Firstly, the two-dimensional transform is given by:

$$
\mathcal{L}\{\varphi(x, y)\}=\int_{0}^{\infty} \int_{0}^{\infty} \varphi(x, y) \cdot e^{-x \xi_{1}-y \xi_{2}} d x d y
$$

Also, as we have taken advantage of several times:

$$
\omega^{\prime}=u_{x}-i u_{y}=v_{y}+i v_{x} .
$$

From these, we derived the Laplace transform of first partial derivatives of functions:

$$
\begin{gathered}
\mathcal{L}\left\{f_{x}\right\}=\xi_{1} \mathcal{L}\{f(x, y)\}-\mathcal{L}_{y}\{f(0, y)\} \\
\mathcal{L}\left\{f_{y}\right\}=\xi_{2} \mathcal{L}\{f(x, y)\}-\mathcal{L}_{x}\{f(x, 0)\} .
\end{gathered}
$$

In order to directly tackle higher-order equations, we first need to obtain a general form for the Laplace transform of (Eq 3.8):

$$
\omega^{(n)}(z)=\left(\frac{1}{2}\right)^{n} \cdot \sum_{l=0}^{n}\left[\binom{n}{l}(-i)^{l}\left(u^{(n-l, l)}+i v^{(n-l, l)}\right)\right]
$$

which requires a generalization of the Laplace transform of higher-order partial derivatives, both strictly in $x, y$, and all combinations of both $x$ and $y$. For this discussion, assume that the functions here are continuous, differentiable, and all of its partial derivatives are also continuous, allowing us to use Clairauts theorem for mixed partial derivatives.

Also, keep in mind that the notation $\left.f_{x}(0, y) \equiv \frac{\partial f(x, y)}{\partial x}\right|_{x \rightarrow 0}$ and not $\frac{\partial}{\partial x}[f(0, y)]$; trust me, I am aware that the later is equal to 0 . For the same-direction higher-order partial directions, induction can show that:

$$
\begin{align*}
\mathcal{L}\left\{f_{x}\right\} & =\xi_{1} \mathcal{L}\{f\}-\mathcal{L}_{y}\{f(0, y)\}, \\
\mathcal{L}\left\{f_{x x}\right\} & =\xi_{1} \mathcal{L}\left\{f_{x}\right\}-\mathcal{L}_{y}\left\{f_{x}(0, y)\right\} \\
& =\xi_{1}^{2} \mathcal{L}\{f\}-\xi_{1} \mathcal{L}_{y}\{f(0, y)\}-\mathcal{L}\left\{f_{x}(0, y)\right\},  \tag{3.50}\\
& \vdots \\
\mathcal{L}\left\{f^{(j, 0)}\right\} & =\xi_{1}^{j} \mathcal{L}\{f\}-\sum_{l=0}^{j-1} \xi_{1}^{l} \mathcal{L}_{y}\left\{f^{(j-l-1,0)}(0, y)\right\} .
\end{align*}
$$

Similarly, one can derive the $y$ equivalent relation to be:

$$
\begin{equation*}
\mathcal{L}\left\{f^{(0, k)}\right\}=\xi_{2}^{k} \mathcal{L}\{f\}-\sum_{l=0}^{k-1} \xi_{2}^{l} \mathcal{L}_{x}\left\{f^{(0, k-l-1)}(x, 0)\right\} . \tag{3.51}
\end{equation*}
$$

For mixed partials, we can begin by analyzing $f^{(1,1)}$ :

$$
\begin{align*}
\mathcal{L}\left\{\left(f_{y}\right)_{x}\right\} & =\xi_{1} \mathcal{L}\left\{f_{y}\right\}-\mathcal{L}_{y}\left\{f_{y}(0, y)\right\} \\
& =\xi_{1} \xi_{2} \mathcal{L}\{f\}-\xi_{1} \mathcal{L}_{x}\{f(x, 0)\}-\mathcal{L}_{y}\left\{f_{y}(0, y)\right\},  \tag{3.52}\\
\mathcal{L}\left\{\left(f_{x}\right)_{y}\right\} & =\xi_{2} \mathcal{L}\left\{f_{x}\right\}-\mathcal{L}_{x}\left\{f_{x}(x, 0)\right\} \\
& =\xi_{1} \xi_{2} \mathcal{L}\{f\}-\xi_{2} \mathcal{L}_{y}\{f(0, y)\}-\mathcal{L}_{x}\left\{f_{x}(x, 0)\right\} .
\end{align*}
$$

If Clairaut's theorem applies, we then have two equivalent ways of defining $f^{(1,1)}$. But before proceeding into the generalization, if Clairaut doesn't hold true, then there will be $\binom{j}{k}$ ways of writing and nonetheless calculating the different variations of $\mathcal{L}\left\{f^{(j, k)}\right\}$. Let's assume that these conditions do hold true, allowing us to reorder derivatives in whatever direction one wants. This implies that we can begin with (Eq 3.50) and apply $k$ of the $y$ derivatives, or start with (Eq 3.51) and apply $j$ of the $x$ derivatives.

We can then substitute ( Eq 3.51 ) into ( Eq 3.50 ) to obtain:

$$
\begin{equation*}
\mathcal{L}\left\{f^{(j, k)}\right\}=\xi_{1}^{j} \xi_{2}^{k} \mathcal{L}\{f\}-\xi_{1}^{j} \sum_{l=0}^{k-1} \xi_{2}^{l} \mathcal{L}_{x}\left\{f^{(0, k-l-1)}(x, 0)\right\}-\sum_{l=0}^{j-1} \xi_{1}^{l} \mathcal{L}_{y}\left\{f^{(j-l-1, k)}(0, y)\right\} \tag{3.53}
\end{equation*}
$$

or one can substitute (Eq 3.50) into (Eq 3.51) to obtain:

$$
\begin{equation*}
\mathcal{L}\left\{f^{(j, k)}\right\}=\xi_{1}^{j} \xi_{2}^{k} \mathcal{L}\{f\}-\xi_{2}^{k} \sum_{l=0}^{j-1} \xi_{1}^{l} \mathcal{L}_{y}\left\{f^{(j-l-1,0)}(0, y)\right\}-\sum_{l=0}^{k-1} \xi_{2}^{l} \mathcal{L}_{x}\left\{f^{(j, k-l-1)}(x, 0)\right\} \tag{3.54}
\end{equation*}
$$

Remember that our goal, in the end, is to solve for $\mathcal{L}\{f\}$, so everything else in (Eq 3.53-54) should be given to us in some form. If we choose to focus on equation (Eq 3.53) we mandate the following things be given to us:

- $\Sigma_{1}$ mandates $f^{(0,0:(k-1))}(x, 0)$.
- $\Sigma_{2}$ mandates $f^{(0:(j-1), k)}(0, y)$.

From here we can take the two-dimensional Laplace transform of (Eq 3.8) and replace $\mathcal{L}\left\{u^{(n-l, l)}\right\}$ and $\mathcal{L}\left\{v^{(n-l, l)}\right\}$ with either (Eq 3.53) or (Eq 3.54) with the replacement $j \rightarrow(n-l)$ and $k \rightarrow(l)$. Thus for a general $n^{\text {th }}$-order differential equation of the form:

$$
\omega^{(n)}=f\left(z, \omega^{\prime}, \omega^{\prime \prime}, \ldots, \omega^{(n-1)}\right)
$$

this gives us a general representation for the left-hand side. One can then apply the Laplace transform to the right-hand side and use whatever identities are available to allow us to equate real-and imaginary parts, until we obtain a system of the form:

$$
\begin{gathered}
\mathcal{L}\{u\}=F\left(\xi_{1}, \xi_{2}\right) \\
\mathcal{L}\{v\}=G\left(\xi_{1}, \xi_{2}\right),
\end{gathered}
$$

thus allowing us to invert to obtain $\omega=u+i v$. One could, of course, approach this via first-order systems as we did previously.

### 3.5 A Differential Approach to Contour Integration

Before concluding this chapter with some side examples to think about, it is to note that the existence and uniqueness theorem for higher-order equations falls as a direct result of first-order system transformations. That is, since each of the first-order equations of the system have a unique solution, its corresponding higher-order equation also has a unique solution. This section will focus on (Eq 3.9):

$$
\frac{2 \pi i}{k!} \omega^{k}(z)=\int_{\gamma} \frac{\omega(\zeta)}{(\zeta-z)^{k+1}} d \zeta=\frac{\pi i}{2^{k-1}} \cdot \sum_{l=0}^{k} \frac{(-i)^{l}}{l!(k-l)!}\left(u^{(k-l, l)}+i v^{(k-l, l)}\right), k \in \mathbb{N},
$$

which is a result that arouse from the higher-order equation generalization and the Cauchy integral equation. Recall, the function $\omega$ must be analytic inside and on the boundary of the simple closed positively oriented contour $\gamma$, where $z \in \gamma$.

Suppose that we are interesting in finding the following contour integral:

$$
\begin{equation*}
I=\int_{\gamma} \frac{e^{z}}{(z-2)^{2}} d z, \quad \gamma:|z|=9 . \tag{3.55}
\end{equation*}
$$

Note here that $2 \in \gamma$ and for $\omega(\zeta)=e^{\zeta}$ and $k=1$, this integral is of the form (Eq 3.9). Therefore for some $F$ :

$$
\begin{align*}
F(z):=\int_{\gamma} \frac{e^{\zeta}}{(\zeta-z)^{2}} d \zeta & =\pi i \cdot\left[\left(u_{x}+i v_{x}\right)-i\left(u_{y}+i v_{y}\right)\right]  \tag{3.56}\\
& =2 e^{x} \pi i[\cos (y)+i \sin (y)]
\end{align*}
$$

Since $I=F(2+0 i)$, we evaluate $F$ at $x=2, y=0$. Therefore:

$$
\begin{equation*}
\int_{\gamma} \frac{e^{z}}{(z-2)^{2}} d z=2 e^{2} \pi i \tag{3.57}
\end{equation*}
$$

One can easily verify that $\operatorname{Res} f(z)=e^{2}$, so this result matches Cauchy's Residue Theorem.

To satisfy curiosity, let us do another example that is a tad bit more involved. Consider the contour integral:

$$
\begin{equation*}
A=\int_{\gamma} \frac{5 z^{2}+2 z+1}{(z-i)^{3}} d z, \quad \gamma:|z|=2 . \tag{3.58}
\end{equation*}
$$

Note here that the singularity $z=i$ lies within $\gamma$, and the singularity has order 3. The reason I note these things is that the order must be greater than or equal to 1 for the corresponding derivative to hold. If one wants to consider $k \in \mathbb{R}$ instead of $k \in \mathbb{N}$, then one can perform an analytical extension to account for fractional-order derivatives (a discussion for another day). And in terms of the singularity lying within our contour, recall that if the singularity does not lie within our contour $\gamma$, then the contour $\gamma$ can be contracted to a point resulting in $\int_{\gamma}=0$ about that singularity; hence we only consider functions $f$ for which are not holomorphic on the entirety of $\gamma$ for discussion purposes. For (Eq 3.58), we approach it the same way by defining a function $G(z)$ and applying (Eq 3.9) with $k=2$. We obtain:

$$
\begin{align*}
G(z):=\int_{\gamma} \frac{5 \zeta^{2}+2 \zeta+1}{(\zeta-z)^{3}} d \zeta & =\frac{\pi i}{2}\left[\frac{1}{2}\left(u_{x x}+i v_{x x}\right)-i\left(u_{x y}+i v_{x y}\right)+\frac{1}{2}\left(u_{y y}+i v_{y y}\right)\right]  \tag{3.59}\\
& =10 \pi i .
\end{align*}
$$

Hence we can see that for any value $a \in \gamma, F(a)=A=10 \pi i$, otherwise if $a \notin \gamma, A=0$; this can readily be verified since $\operatorname{Resf}(z)=5$.

Even though this section is not 'directly' related to this thesis, it is easy to say that I feel that this application/result can be used for a variety of things and definitely will be explored in heavier detail in my near future.

## Chapter 4 TIME-DEPENDENT DIFFERENTIAL EQUATIONS

In this chapter we will discuss time-dependent complex variable differential equations: namely the advection and wave equations. We discuss the finite difference schemes, we discuss/review how to determine the stability of the schemes and also briefly discuss how to interpret the results near singularities.

### 4.1 The Complex Advection Equation

We will begin by noting that we will be considering time as a real variable rather than trying to parse the meaning of complex variable time (e.g. $t=2+2 i$ seconds). Suppose that $\omega$ represents the complex-velocity of a particle at complex-position $z$ at real-time $t$ (hence $\omega=\omega(z, t))$. Let us also assume that the speed of the particle varies with time; i.e. $c=c(t) \in \mathbb{R}$. Hence the one-complex dimensional advection equation can be constructed to be:

$$
\begin{equation*}
\frac{\partial \omega}{\partial t}+c(t) \frac{\partial \omega}{\partial z}=0 \tag{4.1}
\end{equation*}
$$

Since $\omega=u+i v$, we can again rewrite $\omega_{z}$ via the Wirtinger representation and decompose $\omega_{t}$ directly with the time-derivatives of its real and imaginary parts. Equating real and imaginary parts gives us the time-dependent system:

$$
\begin{align*}
& u_{t}+\frac{1}{2} c(t)\left[u_{x}+v_{y}\right]=0  \tag{4.2}\\
& v_{t}+\frac{1}{2} c(t)\left[v_{x}-u_{y}\right]=0
\end{align*}
$$

Here, remember that both $u$ and $v$ are functions of $x, y$ and $t$. Applying the CR equations,
gives us equations for which we can solve through the spatial directions of both $u$ and $v$ :

$$
\begin{gather*}
u_{t}=-c(t) u_{x} \quad \text { and } \quad u_{t}=-c(t) v_{y}  \tag{4.3}\\
v_{t}=c(t) u_{y} \quad \text { and } \quad v_{t}=-c(t) v_{x}
\end{gather*}
$$

We will first note that if we use a forward difference approximation for $u_{t}$ and central difference for $u_{x}$, write out the scheme, and run the results and realize 'the method is unstable.' This issue is discussed thoroughly in [25] and [23]; we will thus take the forward-time centralspace approach and reference the reader to the aforementioned leader for more details on why this approach is prefered/necessary. Take for example the $u_{t}, u_{x}$ equation of ( Eq 4.3 ). This will give us:

$$
\begin{equation*}
u_{n+1}^{j, k}=u_{n}^{j, k}-\frac{c(t) \Delta t}{2 h}\left(u_{n}^{j+1, k}-u_{n}^{j-1, k}\right) \tag{4.4}
\end{equation*}
$$

For notations, we will use superscripts to denote spatial positions and subscripts to denote time positions. The Lax method replaces the $u_{n}^{j, k}$ term with the spatial average at that point to cure the instability issue, which turns (Eq 4.4) into:

$$
\begin{equation*}
u_{n+1}^{j, k}=\frac{1}{2}\left(u_{n}^{j+1, k}+u_{n}^{j-1, k}\right)-\frac{c(t) \Delta t}{2 h}\left(u_{n}^{j+1, k}-u_{n}^{j-1, k}\right) . \tag{4.5}
\end{equation*}
$$

One then would desire to have better accuracy by applying a central difference scheme via a central-order approximation for $u_{t}$, but one would have to again investigate the stability of the scheme. But again, we refer to the references cited for more details on this since it has been investigated thoroughly by others. For now, we will move forward with (Eq 4.5) and allow our $\Delta t$ and $h$ magnitude dictate the accuracy of our methods. For an exact solution to our PDE (Eq 4.1) to exist, we need to prescribe an initial condition $(t=0)$ to be:

$$
\begin{equation*}
\omega(z, 0):=f(z) \tag{4.6}
\end{equation*}
$$

and a time-varying boundary condition $\left(z=z_{0}\right)$ to be:

$$
\begin{equation*}
\omega\left(z_{0}, t\right):=g(t) \tag{4.7}
\end{equation*}
$$

The initial condition (Eq 4.6) will allow us to know all points $u_{0}^{j, k}+i v_{0}^{j, k}$ for all points $(x, y)$ inside and outside of $\mathcal{D}$, so the need of constructing ghost cells and so on is irrelevant in this case, as the entire right-hand of ( Eq 4.5 ) depends on $\omega_{0}$ in order to determine $\omega_{n}$ for all $n>0$. In terms of the boundary condition, the prescription is dependent on the problem that is being modeled/considered. For example, if you want to model a water wave in a pool, then a reflective boundary condition; whereas if you want to model a sound wave in a sound-proof room, a absorbing boundary condition may be appropriate. Numerically, some sort of boundary behavior is desired in order to limit the domain space $\mathcal{D}$ to something computationally practical. In a similar fashion to equation (Eq 4.5):

$$
u_{n+1}^{j, k}=\frac{1}{2}\left(u_{n}^{j+1, k}+u_{n}^{j-1, k}\right)-\frac{c(t) \Delta t}{2 h}\left(u_{n}^{j+1, k}-u_{n}^{j-1, k}\right) .
$$

which was built from the $u_{t}, u_{x}$ equation of ( Eq 4.3 ), one can derive a similar equation for $v_{n+1}$ from the $v_{t}, v_{x}$ equation of $(\mathrm{Eq} 4.3)$ to be:

$$
\begin{equation*}
v_{n+1}^{j, k}=\frac{1}{2}\left(v_{n}^{j+1, k}+v_{n}^{j-1, k}\right)-\frac{c(t) \Delta t}{2 h}\left(v_{n}^{j+1, k}-v_{n}^{j-1, k}\right) . \tag{4.8}
\end{equation*}
$$

For discussion purposes, consider a complex Gaussian initial condition:

$$
f(z)=a \cdot e^{-k \cdot z^{2}}, a, k \in \mathbb{R}, \frac{d c}{d t}=0
$$

and let us solve the advection equation on the computational domain $\mathcal{D}=[-5,5] \times[-5,5]$ with $h=\Delta t=0.05$ in addition to prescribing an absorbing boundary. As seen in (Figure 4.1), the solution appears to behave nicely and tends to dissipate into the boundary as


Figure 4.1: Complex Advection Equation (Re/Im)w/ $\Delta t=0.05$ at $t=5$ and $t=15$


Figure 4.2: Complex Advection Equation (Re/Im)w/ $\Delta t=0.2$ at $t=1.2$ and $t=4$
$t \rightarrow \infty$ as designed. In some mathematical models, it is useful to track the model into the near (or far) distant future. One approach to this desire is to re-scale the model to be, say, in terms of hours rather than seconds; this requires modification of the equation terms, but is not highly difficult. Another approach is to just increase the value of $\Delta t$ to something higher. For example, if $\Delta t=1$ corresponds to every frame showing concurrent seconds, $\Delta t=60$ will correspond to every frame showing concurrent minutes; this does not require modification of the model and is very easy to code. As seen in (Figure 4.2), increasing $\Delta t$ from 0.05 to 0.2 creates a slight disturbance in the solution, which does not exist in smaller values of $\Delta t$. This ties into the stability of the finite difference scheme.

We can determine the values of $\Delta t$ that will cause our finite difference scheme to converge/diverge by performing Von Neumann stability analysis; let us take (Eq 4.5) to illustrate the idea. We will first define a (time varying) constant $r$ to be:

$$
\begin{equation*}
r:=\frac{c(t) \Delta t}{h} . \tag{4.9}
\end{equation*}
$$

This number is sometimes referred to as the Courant-Freidrichs-Lewy (CFL) number. Our equation then becomes:

$$
\begin{equation*}
u_{n+1}^{j, k}=\frac{1}{2}\left(u_{n}^{j+1, k}+u_{n}^{j-1, k}\right)-\frac{r}{2} \cdot\left(u_{n}^{j+1, k}-u_{n}^{j-1, k}\right) . \tag{4.10}
\end{equation*}
$$

Since our finite difference methods only have spatial derivatives in one direction, we can treat the $k$ index in (Eq 4.10) as if it is not there since the error in the solution calculation will not propagate into the $y$ direction. We can then expand the spatial variation of the error, $\epsilon$, into a finite Fourier series in the one-dimensional $(x)$ interval $L$ pertaining to our computational domain $\mathcal{D}$. We will include the $y$ portion of the Fourier expansion for general. We can write this as:

$$
\begin{equation*}
\epsilon_{m}(x, t)=e^{\sigma t} e^{i\left(j_{m} x+k_{m} y\right)} \tag{4.11}
\end{equation*}
$$

where $j_{m}=\frac{\pi m}{L}$ and $k_{n}=\frac{\pi n}{L}$ are the wave numbers with $m, n=1,2, \ldots, \frac{L}{h}$ and $\sigma$ a constant. In the case that $\mathcal{D}$ is not a square domain, one an define $L_{x}$ and $L_{y}$ if desired. Hence we can
rewrite each of our finite difference terms as:

$$
\begin{align*}
u_{n}^{j, k} & =e^{\sigma t} e^{i\left(j_{m} x+k_{n} y\right)} \\
u_{n+1}^{j, k} & =e^{\sigma(t+\Delta t)} e^{i\left(j_{m} x+k_{n} y\right)} \\
u_{n}^{j+1, k} & =e^{\sigma t} e^{i\left(j_{m}(x+h)+k_{n} y\right)}  \tag{4.12}\\
u_{n}^{j-1, k} & =e^{\sigma t} e^{i\left(j_{m}(x-h)+k_{n} y\right)} .
\end{align*}
$$

We can then rewrite ( $\operatorname{Eq} 4.10$ ) into its Fourier form to be:

$$
\begin{align*}
e^{\sigma(t+\Delta t)} e^{i\left(j_{m} x+k_{n} y\right)}= & \frac{1}{2}\left(e^{\sigma t} e^{i\left(j_{m}(x+h)+k_{n} y\right)}+e^{\sigma t} e^{i\left(j_{m}(x-h)+k_{n} y\right)}\right)-  \tag{4.13}\\
& \frac{r}{2} \cdot\left(e^{\sigma t} e^{i\left(j_{m}(x+h)+k_{n} y\right)}-e^{\sigma t} e^{i\left(j_{m}(x-h)+k_{n} y\right)}\right) .
\end{align*}
$$

As mentioned before, our finite difference equation does not contain discretizations of the $y$-derivative, so we can divide both sides of (Eq 4.13) by $e^{k_{n} y}$ and also $e^{\sigma t}$. This gives us:

$$
e^{\sigma \Delta t} e^{i j_{m} x}=\frac{1}{2}\left(e^{i j_{m}(x+h)}+e^{i j_{m}(x-h)}\right)-\frac{r}{2} \cdot\left(e^{i j_{m}(x+h)}-e^{i j_{m}(x-h)}\right) .
$$

Simplifying our exponential expressions gives us:

$$
\begin{equation*}
e^{\sigma \Delta t}=\frac{1}{2}\left(e^{i j_{m} h}+e^{-i j_{m} h}\right)-\frac{r}{2} \cdot\left(e^{i j_{m} h}-e^{-i j_{m} h}\right) . \tag{4.14}
\end{equation*}
$$

We can rewrite the exponential terms of ( Eq 4.14 ) in terms of sin and cos to be:

$$
e^{\sigma \Delta t}=\cos \left(j_{m} h\right)-i r \sin \left(j_{m} h\right) .
$$

The Lax-equivalence theorem states that a necessary and sufficient condition for the error
to be bounded is that $|G| \leq 1$, where

$$
\begin{align*}
G & =\frac{\epsilon_{n+1}^{j, k}}{\epsilon_{n}^{j, k}} \\
& =\frac{e^{\sigma(t+\Delta t)} e^{i\left(j_{m} x+k_{n} y\right)}}{e^{\sigma(t)} e^{i\left(j_{m} x+k_{n} y\right)}}  \tag{4.15}\\
& =e^{\sigma \Delta t} .
\end{align*}
$$

Therefore, the error of the finite difference scheme (Eq 4.10) will be bounded if and only if:

$$
\begin{equation*}
\left|\cos \left(j_{m} h\right)-i r \sin \left(j_{m} h\right)\right| \leq 1 \tag{4.16}
\end{equation*}
$$

This is equivalent to the inequality equation:

$$
\begin{equation*}
\left(r^{2}-1\right) \sin ^{2}\left(j_{m} h\right) \leq 0 \tag{4.17}
\end{equation*}
$$

and since $\sin ^{2}\left(j_{m} h\right) \geq 0$ for all $j_{m} h$, hence we get the condition that $r^{2} \leq 1$ in order for stability to be achieved. Therefore,

$$
\begin{equation*}
\Delta t \leq \frac{h}{c(t)} \tag{4.18}
\end{equation*}
$$

Note also that this is the same result as in the real-variable case. Therefore, if you choose the same numerical values as we did in our simulation of the advection equation (namely $c(t)=0.4$ and $h=0.05$ ), then the finite difference method ( Eq 4.10 ) will be stable in its solution if and only if $\Delta t \leq 0.125$, which explains why $\Delta t=0.2$ caused an instability in our solution.

The last thing to mention in terms of the solution method for the advection equation, recall
that from (Eq 4.3), we chose to use the equations:

$$
u_{t}=-c(t) u_{x}, \quad v_{t}=-c(t) v_{x}
$$

to obtain our numerical solutions $u, v$. Note here that these two equations are the same aside from real-part/imaginary-part difference, hence the CFL condition is the same for both to determine the stability of the solutions for both $u, v$. An alternative would be to use the system:

$$
u_{t}=-c(t) v_{y}, \quad v_{t}=c(t) v_{y} .
$$

It is easy to see that the CFL condition is the same for these two equations (due to the definition of $|\cdot|$ ), and the numerical solutions of the two systems are comparable as was the BT and LR methods of the previous chapters.

### 4.2 The Complex Wave Equation

In continuation of the previous section, we proceed to a second order version of the advection equation, namely the wave equation:

$$
\begin{equation*}
\frac{\partial \omega}{\partial t}=c(t) \cdot \frac{\partial^{2} \omega}{\partial z^{2}} \tag{4.19}
\end{equation*}
$$

We again rewrite $\omega_{t}$ as $u_{t}+i v_{t}$. For the $\omega_{z z}$ term, we will use (Eq 3.3) to rewrite it in terms of $x, y$ partial derivatives. Therefore:

$$
\begin{equation*}
u_{t}+i v_{t}=\frac{1}{4} c(t) \cdot\left[\left(u_{x x}-2 i u_{x y}-u_{y y}\right)+i\left(v_{x x}-2 i v_{x y}-v_{y y}\right)\right] . \tag{4.20}
\end{equation*}
$$

Equating real and imaginary parts gives us the system:

$$
\begin{align*}
u_{t} & =\frac{1}{4} c(t)\left[u_{x x}-u_{y y}+2 v_{x y}\right]  \tag{4.21}\\
v_{t} & =\frac{1}{4} c(t)\left[v_{x x}-v_{y y}-2 u_{x y}\right]
\end{align*}
$$

For $u_{t}$ and $v_{t}$, we will use a forward difference approximation as before. For $u_{x x}, u_{y y}$ (and similarly for $v_{x x}, v_{y y}$ ), we will use the second order central difference approximation:

$$
\begin{align*}
& u_{x x}: \approx \frac{1}{h^{2}}\left(u_{n}^{j+1, k}-2 u_{n}^{j, k}+u_{n}^{j-1, k}\right),  \tag{4.22}\\
& u_{y y}: \approx \frac{1}{h^{2}}\left(u_{n}^{j, k+1}-2 u_{n}^{j, k}+u_{n}^{j, k-1}\right) .
\end{align*}
$$

For the $u_{x y}=u_{y x}$ (and similarly for $v_{x y}=v_{y x}$ ) term, we have:

$$
\begin{align*}
& u_{x y}: \approx \frac{1}{h^{2}}\left(u_{n}^{j+1, k+1}-u_{n}^{j+1, k}-u_{n}^{j, k+1}+u_{n}^{j, k}\right),  \tag{4.23}\\
& v_{x y}: \approx \frac{1}{h^{2}}\left(v_{n}^{j+1, k+1}-v_{n}^{j+1, k}-v_{n}^{j, k+1}+v_{n}^{j, k}\right) .
\end{align*}
$$

This creates one method of solving the wave equation. To make things more simple for analysis, recall that $u, v$ satisfy the Cauchy-Riemann equations. Therefore:

$$
u_{x y}=-v_{x x} \quad \text { and } \quad v_{x y}=u_{x x} .
$$

This allows us to transform (Eq 4.21) into:

$$
\begin{align*}
& u_{t}=\frac{1}{4} c(t)\left[3 u_{x x}-u_{y y}\right]  \tag{4.24}\\
& v_{t}=\frac{1}{4} c(t)\left[3 v_{x x}-v_{y y}\right] .
\end{align*}
$$

Hence the $u_{t}$ and $v_{t}$ finite-difference equations for ( Eq 4.24 ) is given by:

$$
\begin{align*}
& u_{n+1}^{j, k}=u_{n}^{j, k}+\frac{c \Delta t}{4 h^{2}}\left(3 u_{n}^{j+1, k}-u_{n}^{j, k+1}-4 u_{n}^{j, k}+3 u_{n}^{j-1, k}-u_{n}^{j, k-1}\right)  \tag{4.25}\\
& v_{n+1}^{j, k}=v_{n}^{j, k}+\frac{c \Delta t}{4 h^{2}}\left(3 v_{n}^{j+1, k}-v_{n}^{j, k+1}-4 v_{n}^{j, k}+3 v_{n}^{j-1, k}-v_{n}^{j, k-1}\right)
\end{align*}
$$

Even though (Eq 4.24) and (Eq 4.21) are analytically equivalent, the finite difference methods could potentially gain different numerical results. Therefore, using (Eq 4.22) and (Eq 4.23) to obtain a finite-difference method for (Eq 4.21), we obtain the following finite-difference equations:

$$
\begin{align*}
u_{n+1}^{j, k} & =u_{n}^{j, k}+\frac{c \Delta t}{4 h^{2}}\left(u_{n}^{j+1, k}-u_{n}^{j, k+1}+u_{n}^{j-1, k}-u_{n}^{j, k-1}+2 v_{x y}\right),  \tag{4.26}\\
v_{n+1}^{j, k} & =v_{n}^{j, k}+\frac{c \Delta t}{4 h^{2}}\left(v_{n}^{j+1, k}-v_{n}^{j, k+1}+v_{n}^{j-1, k}-v_{n}^{j, k-1}-2 u_{x y}\right),
\end{align*}
$$

where $v_{x y}$ and $u_{x y}$ are represented by (Eq 4.23). For the Von-Neumann stability analysis, our CFL number is given by:

$$
\begin{equation*}
r:=\frac{c(t) \Delta t}{h^{2}} . \tag{4.27}
\end{equation*}
$$

To reduce the instability of the finite-difference scheme, we will again apply the Lax Method for the $u_{n}^{j, k}$ term in each of the equations. For the implementation, finite-difference system ( Eq 4.25 ) is uncoupled and can be split on two different computational threads, whereas the finite-difference system (Eq 4.26) cannot be 'easily' parallelized.

One of the last things to mention in regards to these approaches is one of the downsides to finite difference methods. Let us assume that our initial condition, $f(z)$, has one or more singularities in the computational domain; for example $f(z)=\frac{1}{1+z^{2}}$. Therefore, if one is use finite difference methods for these type of problems (impulse problems such as $f(z)=\delta(z)$ ) must by taken with caution as the singularity undefined-ness ripples outwards from the original singularities. This may not be the case for both $\operatorname{Re}(\omega)$ and $\operatorname{Im}(\omega)$, but the numerical solution will provide such a conclusion as shown in the following set of figures.


Figure 4.3: Complex Wave Equation Finite-Difference Results Around Singularities

To verify our results, consider the wave equation with $c=1$ :

$$
\begin{equation*}
\frac{\partial \omega}{\partial z}=\frac{\partial^{2} \omega}{\partial z^{2}} \tag{4.28}
\end{equation*}
$$

It is easy to see that the function:

$$
\begin{equation*}
\omega(z, t)=e^{-t / 10} \sinh \left(\frac{i}{\sqrt{10}} z\right) \tag{4.29}
\end{equation*}
$$

satisfies (Eq 4.28), which will allow us to verify our solution numerically. Therefore, it is necessary to prescribe the initial condition for ( Eq 4.28 ) to be:

$$
\begin{equation*}
f(z)=\sinh \left(\frac{i}{\sqrt{10}} z\right) . \tag{4.30}
\end{equation*}
$$

As an easy exercise, we can find the real/imaginary decomposition of our test solution to be:

$$
\begin{align*}
& u(x, y, t)=-e^{-t / 10} \cos \left(\frac{x}{\sqrt{10}}\right) \sinh \left(\frac{y}{\sqrt{10}}\right) \\
& v(x, y, t)=e^{-t / 10} \sin \left(\frac{x}{\sqrt{10}}\right) \cosh \left(\frac{y}{\sqrt{10}}\right) . \tag{4.31}
\end{align*}
$$

A time frame of the real/imaginary parts of the solution together with the Riemann surface can be seen in Figure 4.4.

To compare the numerically obtained solution, we can compare the exact solution to the numerical solution at varying time steps through the simulation at sub-domains that are not affected by the boundary condition implemented. At time $t=0$, the numerical and theoretical results are trivially equal, so for our numerical comparison, we have taken three time steps after $t=0$ and taken the $L_{2}$ norm of difference of solutions with the ideal case that the norm is equal to zero; the results are given in Figure 4.5 (vertical axis is the $L_{2}$ error, and horizontal axis is the number of time steps taken). As evident from the analytical solution, as $t \rightarrow \infty$, the true solution $\omega \rightarrow 0$, which both the analytical and numerical solutions converge to at about the same time.


Figure 4.4: Complex Wave Equation Finite-Difference Exact Solution Time Frame


Figure 4.5: Time Series of $L_{2}$ Error in Solutions for the Wave Equation

## Chapter 5 CONCLUSIONS AND FINAL REMARKS

In this final chapter, we will summarize the work that has been discussed in this thesis and some of the contributions that has been laid down. We will also discuss some of the problems that we are considering for future applications and investigation.

### 5.1 Summary of Results

The first chapter of this thesis is primarily dedicated to literature review and discussions of things that has already been done, but there are a couple things that can be discussed in more detail and be applied to other potential problems. As the majority of this thesis is focused around the perspective of solving complex-variable problems in the perspective of two-real-variable problems, the acquisition of the real and imaginary parts of a complex function, $\omega(z)$, is/can be of vital importance. For a complex function, $\omega(z)$, one can easily numerically obtain the real and imaginary parts, $u$ and $v$; simply substitute various numbers for $x, y$ into $\omega(x+i y)$, and allow for whatever numerical software to calculate the real and imaginary parts of it; to some, this isn't enough. Having the ability to quickly obtain the analytical representation of $u(x, y)$ and $v(x, y)$ may be of importance to some, but due to the nature of algebraic structures, a closed form may not exist (similar to elementary primitives).

The second chapter is, the heart of this thesis, as it presents the main idea and goals of what we aimed to achieve. The main results that we have taken advantage of are:

- The Cauchy-Riemann Equations,
- The conversion of $\partial_{z}$ into $\partial_{x}$ and $\partial_{y}$ (Wirtinger).

Equating real and imaginary parts proved that solving a single-complex-variable ordinary (or equivalently partial) differential equation is equivalent to solving a two-real-variable system of partial differential equations. In the spatial domain, we considered to different directions:
the left-to-right (LR) and bottom-to-top (BT) methods. We realized that even though the two methods do not, unsurprisingly, produce the same numerical results, they do converge to the same solution.

We also proved the existence and uniqueness theorem for first-order complex initial value problems and illustrated that the iteration method still can be applied for numerical purposes. We have not seen much extensive applications of multi-variable Laplace transforms in the literature, so we felt that it was necessary to approach it in an elementary approach that would be useful for other people who may be interested in investigating analytical methods.

In the third chapter, we derived a generalized real-imaginary decomposition of a complexvalued function $\omega$ that can be used to translate a higher-order complex differential equation into a real-valued partial differential system. We demonstrated that this is not algebraically efficient, and should only be approached if the problem was extremely general or a particular narrowed down problem for investigation. We demonstrated that the first-order system approach is still a viable alternative that can be used to solve non-linear complex-variable higher-order equations. As a side result, from the generalization of $D^{\alpha}$, we were able to find a partial-derivative alternative for evaluating contour integrals.

In the fourth chapter, we demonstrated that adding time to the complex variable problem was a trivial extension if we assumed that time itself was real valued. If we break this assumption, then the complexity may again arise, but the same approach can be used for the spatial derivatives.

### 5.2 Future Considerations

There are some theoretical investigations that one may pursue in response to some of the numerical approaches discussed here. In regards to the LR and BT methods, we hope
to determine a necessary condition as to which method will be the best approximation to the solution under some assumptions about the original equation or initial conditions. A simple modification that one can propose is taking the average of the LR and BT methods to construct a "mid-point" like method. As this is a very easy modification to the code, we did not have significant results to prove that it was any more favorable than the other, so we chose not to include it. As we did not consider the theoretical aspects of convergence (as it was not my focus), a guarantee of its order convergence being higher than LR or BT is no expected based on the side results that were obtained. From my knowledge, this is most likely due to the higher-dimensional nature of the problem in consideration.

Another problem that we are curious to investigate in the future is multi-variate differential equations. Take for example the real-variable differential equation $y^{\prime}=\arctan (x)$. It is typically assumed that the branch considered for this is the one whose range is $\frac{1}{2}(-\pi, \pi)$. If this is true, then the problem is asking "what functions derivative has this behavior on this interval". Choosing another branch of inverse tangent will have a different but similar goal, but the solution would itself be different. Branches are not often considered in the real-variable case, even though there are several real-valued functions that have multiple branches, namely the inverses of $y=x^{2}, y=\sin (x)$, the solution of $y$ for $y e^{y}=x$ (namely the Lambert W function) to name a few. Branches come up more often in the realm of complex analysis due to the "new" (but not really) nature of the logarithmic and exponential functions, mainly due to the multi-variate nature of $f(z)=\arg (z)$.

Also in the second chapter, we illustrated the iterative method for solving complex initial value problems. As this method is effective, it can be computationally expensive (by hand). For numerical purposes, one would have to have a symbolic system to execute the calculation of the indefinite integrals. For the existence and uniqueness theorem, certain properties of $f(z, \omega)$ must be met (analyticity being one). But for some functions, such as $\operatorname{Re}(z)$ and $\bar{z}$
to name two trivial functions, this criteria is not met; this does not necessarily imply that a (local) solution exists for the differential equation, so investigating these "special" problems would be useful to satisfy curiosity.

We chose to use Laplace transforms to analyze the analytical solution of the complex differential equations. This method requires knowledge of the real-imaginary decomposition of functions to find the two-variable Laplace transform of. The interest of integral transforms has disappeared from the primary interest of research in the past few decades, but we still feel that it poses some value still in the modern day (as so many applications are still in use today). A couple problems that we do plan on investigating in regards to Laplace transform solutions of differential equations are the following points:

- Numerical construction of the Laplace transform of a real and/or complex valued function.
- Constructing an numerically efficient inverse Laplace transform of a transformed function. As was discussed in chapter 4, a partial-differential method was derived that may hold promise for this goal.
- From a initial value problem, we able to construct the boundary of the solution in order to use Laplace transforms for the solution. Some have referred to these problems as "immersed boundary problems", but we feel that more work can be done in this field.

In the third and fourth chapter, we discussed applications to higher-order and time-varying differential equations in the complex spatial domain. We feel that so much work can be done in this realm, both in the real and complex sense. One of the primary future goals that we want to consider is developing a spectral method for solving a complex-variable equation. We believe that this approach will be more of a tensor algebra problem, but nonetheless it
will benefit the numerical realm as spectral methods are one of the top preferred methods of solving differential equation problems. We also plan to investigate the partial-differential approach to evaluating contour integrals; we hope to develop a necessary and sufficient condition that will guarantee that this method will work. This will allow numerical contour integration to easily see an advantage. The generalization of the complex differential operator can also be extended to the fractional calculus sense in the traditional manner as was the real differential operator, so we also plan to investigate the consequences of that extension. Due to the variety of directions one can take, we hope that we gain develop physical applications of these things to fields such as fluid dynamics, quantum physics and computer vision.

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