PLANE AND SOLID GEOMETRY

A SERIES OF MATHEMATICAL TEXTS

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EARLE RAYMOND HEDRICK

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PLANE AND SOLID GEOMETRY

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PREFACE

While the preparation of a textbook on Geometry presents much the same problem whenever undertaken, the arrangement of its details and the balancing of its parts must be expected to change as new educational ideals become established, and as new conditions of society arise.

It is no longer possible, for example, to expect favor for the typical lengthy text of a generation ago. Although such texts proceed in good logical fashion to deduce the traditional theorems one after another, yet they make little or no appeal to the world of common experience; and they present far more material than the student can absorb in the course as it is now usually taught. Neither is the ideal book something so far removed from this traditional type that its chief feature is a large variety of illuminating diagrams drawn from the Arts.

Between these two extremes lies the only proper course to-day, and this the authors have tried diligently to find. The traditional manner of presentation in a logical system is preserved, but logical development is not made the sole purpose of the book. Thus, problems drawn from the affairs of practical life are inserted in considerable number. The function of such problems is not to train the student in the technique of any of the Arts; rather it is to illuminate the geometric facts, and to make clear their importance and their significance.

Geometry is and is likely to remain primarily a cultural, rather than an informational subject. But the intimate connection of Geometry with human activities is evident upon every hand, and constitutes fully as much an integral part of the subject as does its older logical and scholastic aspect.

Teachers of Geometry throughout the country recognize this and appreciate strongly the value of the study, not only as subject-matter in which practice in logical processes is ideally simple and beautiful, but also as an instrument for meeting real human needs. It is certainly most desirable from the standpoint of effective instruction that the *students* also come to realize all of these merits, thus answering for themselves the persistent question as to why the subject is studied. The necessary attention and concentration of mind on the students' part will then become from the very outset easy and natural.

Attention is called to the Introductory chapter. In it the student is acquainted with the use of the ruler and compasses in the more simple construction problems, and is thus brought early face to face with many of the fundamental notions of Geometry. Great care has been exercised at this point that the student may approach the more rigid demands of the deductive side of the subject gradually, rather than plunge into them at the outset. Even in Chapter I, in which strictly deductive work commences, no abrupt change of style occurs, but an easy transition is made into the parts that so often appear foreign and unintelligible to the beginner. Indeed, throughout the book, an effort has been made to soften the austere and unnatural style that has frequently proved a bar to ready comprehension, and to avoid an excess of symbolic characters and technical phrases that do not add to the reasoning.

The book is distinguished by its acceptance of the principle of emphasis of important theorems laid down by the Committee of Fifteen of the National Education Association in their Report.* Thus, theorems of the greatest value and importance are printed in bold-faced type, and those whose importance is considerable are printed in large italics.

* Printed as a separate pamphlet with the *Proceedings* for 1912. Reprinted also in *School Science*, 1911, and in *The Mathematics Teacher*, December, 1912.

The Report just mentioned has been of great assistance, and its principles have been accepted in general, not in a slavish sense but in the broad manner recommended by the Committee itself. A perusal of the Report will give, more fully and accurately than could be done in this brief preface, the considerations which led to the adoption of these principles, in particular, the principle of emphasis upon important theorems, both by the Committee and by the authors of this book.

The authors are indebted also to many another book and to many recent reports and papers for ideas and problems. In the Introduction, the effect of the excellent little book of Godfrey and Siddons * is sufficient to deserve explicit mention.

Detailed description of the principles followed in the Solid Geometry may be omitted, since they are in spirit the same as those of the Plane Geometry. The great excellence of the figures, particularly that of the very unusual and effective 'phantom' halftone engravings in the Solid Geometry, deserves mention. These figures should go far toward relieving the unreality which often attaches to the constructions of Solid Geometry in the minds of students.

W. B. FORD.
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E. R. HEDRICK, EDITOR.

* Godfrey and Siddons, Plane Geometry, Cambridge University Press.

CONTENTS

PLANE GEOMETRY

INTRODUCTION

PART I. DRAWING SIMPLE FIGURES

1. Geometry is the branch of mathematics that deals with space. In it we consider points, lines, triangles, circles, spheres, and so forth. Such questions as the following are considered: When do two triangles of different shapes have the same area? How does the length around a circle compare with the diameter? How can the volume of a sphere be calculated? Geometry not only answers many questions of this kind, but it also presents a systematic study of the general principles involved, and the reasons for all statements are given with care.

2. Ruler and Compasses. In studying geometry it is desirable to draw accurate figures. The student will need a *ruler*, with which to draw (or extend) straight lines; and *a pair of compasses*, with which to draw circles and to transfer lengths from one position to another.

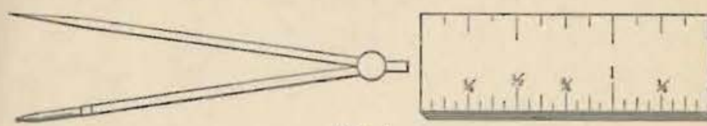


FIG. 1

For distinctness, the curve that forms the circle is often called the **circumference** of the circle. The position of the fixed point of the compasses is called the **center** of the circle. Any portion of the circumference is called an **arc** of the circle. The distance from the center to the circumference, which is the same all the way around the circle, is called the **radius**.

3. Problem 1. *Draw with a ruler and compasses a triangle each of whose sides is one inch long.*

[A triangle is a figure bounded by three straight sides.]

Solution. With a ruler and pencil draw a line and mark two points A and B on it one inch apart.

Around each of these points as a center, with a radius one inch long, draw an arc of a circle above the line AB so that the two arcs cut each other in a third point, which we will call C.

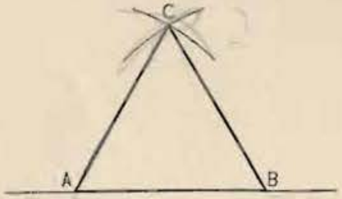


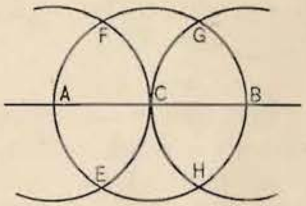
FIG. 2

Connect A with C by a straight line, and connect B and C by another straight line.

The triangle ABC is the required figure; state why each side is one inch long.

EXERCISES

1. Show how to draw a triangle each of whose sides is a given length AB. [HINT. See Problem 1.]



2. Copy the adjoining figure. Draw all the triangles that have equal sides which you can by drawing straight lines connecting pairs of points marked in the figure.

3. Draw in separate positions on heavy paper or on cardboard two triangles, each of whose sides is three inches long. Cut out one of these. Will it fit precisely upon the other one? Will it fit one drawn by another student?

4. Problem 2. *Draw with a ruler and compasses a triangle whose three sides are equal, respectively, to three given lengths.*

Solution. Let a, b, c be the three given lengths. (Fig. 3.)

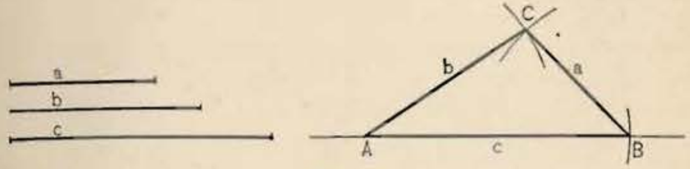


FIG. 3

Draw a line. Mark two points A and B on it so that the distance AB is equal to one of the given lengths, say c. This is done with the compasses.

Around A as center draw an arc of a circle with a radius equal to another of the given lengths, say b.

Around B as center draw another arc with a radius equal to the remaining given length a, and in such a position that the two arcs cut each other.

Let the point where the two arcs cut each other be called C. Draw CA and CB. Then the triangle ABC is the one required.

EXERCISES

1. Draw, with ruler and compasses, a triangle whose three sides are, respectively, two inches, three inches, and four inches in length.

2. Is there any difference between this triangle and one whose sides are, respectively, four inches, two inches, and three inches in length? Draw both triangles separately on cardboard, cut one out, and see if it will exactly fit on the other one. Was it necessary to turn it over?

3. What can you say about any two triangles if the three sides of one are equal, respectively, to the three sides of the other?

5. **Problem 3.** *Draw, by means of ruler and compasses, a perpendicular* to a given straight line at a point in that line.*

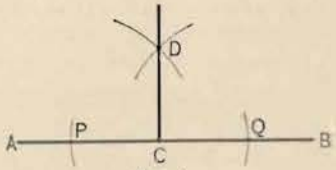


Fig. 4

Solution. Let AB be the given line and C the given point in it.

With C as center, and with any convenient radius, draw arcs of the same circle cutting AB at two points which we will call P and Q.

With a somewhat longer radius than before, draw two arcs of circles above AB with centers at P and Q, being careful that these arcs have the same radius. These two arcs will cut each other at a point that we will call D.

Draw the straight line joining C and D. This is the required perpendicular to AB at C; and there is no other.

EXERCISES

1. Draw with ruler and compasses a perpendicular to a line four inches long at a point one inch from the right-hand end.

2. Draw a triangle with two sides perpendicular to each other, making the perpendicular sides three inches long and four inches long, respectively. Measure the third side.

[The third side should be five inches long. In making the drawing, any lines may be extended as desired.]

* The perpendicular CD can be drawn, of course, with a carpenter's square or a drawing triangle; but the problem is to draw it *with ruler and compasses only*.

The drawing may be tested by folding it in a crease along the line CD; then one end CB of AB should fall on the other end CA of AB. *If it does, CD is said to be* **perpendicular** *to AB at C.*

6. **Problem 4.** *Draw with ruler and compasses a perpendicular to a given straight line from a given point not on that line.*

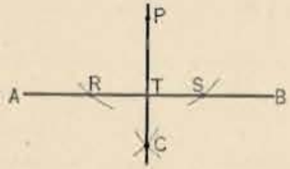


Fig. 5

Solution. Let AB be the given line and let P be the given point not on that line.

With P as center and with any radius long enough to reach past the line AB, draw two arcs of the same circle so as to cut the line AB in two points, which we shall call R and S.

Around the points R and S as centers, with the same radius for both, draw two arcs of circles below AB, and let C be the point where they cut each other.

The straight line joining P and C is the desired perpendicular. Show this by folding the figure on the line PC and convincing yourself that TSB falls along TRA.

EXERCISES

1. Draw a triangle whose three sides are in the ratios 2:3:3. Then draw with ruler and compasses a perpendicular from each corner to the opposite side.

[The accuracy of the drawing can be tested by the fact that these three perpendiculars should meet in one point.]

2. In every drawing of one side of a house, a line is drawn representing the horizontal (or base) line. Show how to draw with ruler and compasses, through any point in the figure, a vertical line, that is, a perpendicular to the base line.

7. Problem 5. *Draw with ruler and compasses an angle equal to a given angle.*

[An **angle** is formed by two portions of straight lines that end at the same point.]

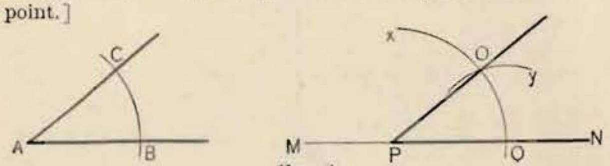


FIG. 6

Solution. Given the angle at A, the problem is to draw an angle equal to A from a point P on another line MN.

With A as center, draw an arc of a circle cutting the lines that form the angle A in the two points B and C.

With P as a center, draw an arc x with the same radius as that just used, cutting MN at Q and extending upwards quite a distance.

With Q as center, and with a radius equal to the distance from B to C, draw an arc y cutting the arc x at a point O.

Then the angle at P formed by the portions of straight lines PQN and PO is equal to the given angle at A, for the two figures will fit each other exactly if one is placed upon the other with A at P and AB along PQN.

EXERCISES

1. Draw two angles that are exactly equal to each other.

[HINT. Draw any angle and then draw another equal to it.]

2. Draw a triangle with one angle equal to the angle in Fig. 6 and with the two sides that form that angle two inches long and three inches long, respectively. Cut out your triangle and compare it with those made by other students. Will it fit exactly on theirs?

3. What can you say of any two triangles if an angle and the two sides that include it in one triangle are equal to the corresponding parts of the other triangle? Why?

8. Problem 6. *Divide a portion of a straight line into two equal parts.*

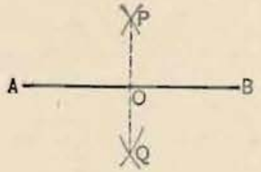


FIG. 7

Solution. Let A and B be two points on a straight line. Draw around A and B as centers, with the same radius, two circles. These circles will cut each other in just two points P and Q if the radius is large enough. Only small arcs of the circles are shown in the figure.

The straight line joining P and Q cuts the line AB at a point O. This point O divides AB into two equal parts AO and OB, for the whole figure may be folded on the line PQ so that O remains at O, and B falls on A.

EXERCISES

1. Show how to divide a line into four equal parts.

2. How can you divide accurately the ruled page of a notebook into two columns of equal width?

3. Show that the line PQ in Fig. 7 is also a perpendicular to the line AB by noting the effect of folding the figure along PQ.

4. Draw any triangle; then find by ruler and compasses the middle point of each of the sides. Connect each corner of the triangle to the middle point of the opposite side by a straight line.

[Do these three new lines meet in one point? The accuracy of the drawing may be tested in this manner.]

9. Problem 7. *Divide a given angle into two equal parts.*

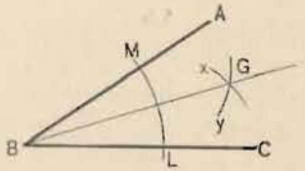


FIG. 8

Solution. Let the given angle be the angle at B between the portions of straight lines BC and BA.

With B as center and with any radius draw a circle that cuts BC at L and BA at M.

With L and M as centers, draw two arcs of circles with the same radius (of any convenient size) so that these two arcs cut each other at a point G.

Then the straight line joining B and G divides the given angle into two equal parts. Convince yourself of this by thinking of folding the figure on the line BG.

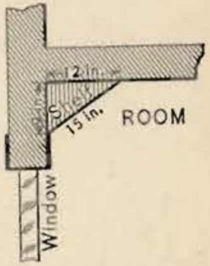
EXERCISES

1. Show how to divide a given angle into four equal parts.

2. Draw any triangle and divide each of its angles into two equal parts. State anything that you notice about the way in which the three new lines meet each other. Try to see whether the same thing happens on another triangle of a different shape. State all your conclusions in one sentence.

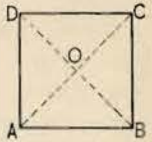
3. Draw two perpendicular lines; then divide the angle between them into two equal parts. State the connection between this problem and the processes that occur, for example, in making the corner of a picture frame, or in mortising a joint. Can you mention any other manufacturing processes in which the same problem arises?

MISCELLANEOUS EXERCISES FOR PART I



1. A shelf is to be fitted into a corner near a window, and is to be triangular. The distance from the corner to the window casing is 9 inches, the shelf is to be 12 inches long on the other wall of the corner, and the edge of the shelf is to be 15 inches long. Are these measurements enough to fix the shape of the shelf? Why? If a carpenter cuts out of a piece of board a triangle that has exactly these lengths of sides, why is it that it will fit in the place? See Problem 2, p. 3, and Ex. 3, p. 3. Will the shelf fit either side up?

2. Draw a triangle with each side just one third the length of the corresponding side of the triangle mentioned in Ex. 1.



3. A square is bounded by four lines of equal length, two of which are perpendicular to each other at each corner. Draw a square whose sides are each two inches long.

[The drawing may be tested by the fact that the two straight lines joining the two pairs of opposite corners should be equal in length.]

4. If the North and South line is shown on any map of a city, show how to draw the East and West line through any point.

5. Show how to draw a triangle with any given angle and with the sides that form that angle of any given lengths. Will two triangles fit each other exactly if made with the same given angle and the same given lengths of the two sides? Must one of the triangles be turned over before they will fit?

6. If two triangles are drawn with two angles of one equal to two angles of the other, respectively, will one necessarily fit the other exactly? What more is needed to *insure* that the triangles shall fit each other exactly?

PART II. THE PRINCIPAL IDEAS USED IN GEOMETRY

10. Solids, Surfaces, Curves, Points. A limited portion of space, such as that bounded by a sphere or a cube, is called a *geometric solid*.

In geometry we have nothing to do with the material of the solid. Thus we would speak of the space occupied by this book as a solid, without thinking that it is made of cloth and paper. Even very soft objects, such as a cloud or a drop of water, would be thought of in geometry only with respect to the form of the portion of space they occupy, and this portion of space would be called a geometric solid, even though the actual object is not solid in the sense of being hard to bend or warp. We often think of the actual object as taken away, and we consider the portion of space that it did or would occupy.

For every solid there is an ideal boundary that separates the solid from the rest of space; this ideal boundary is called a *surface*.

Surfaces are either *curved*, as is the surface of a sphere; or else they are flat in every direction, that is, *plane*. A plane surface is simply called a *plane*. Thus, each of the faces of a cube is a piece of a plane.

When two surfaces cut each other, their common points form a *curve*.

Thus, when a plane cuts a sphere, the curve formed by their common points is a circle. Again, when two planes cut each other, the curve formed is the simplest possible curve — a straight line. As here, we shall often use the word curve, in general, to include straight lines as special curves.

A *point* is that which is common to two curves which cut each other; in particular, two lines cut each other in a point.

11. Ideal Nature of Geometric Things. It should be noticed that the points and curves and surfaces just mentioned are ideal things which cannot be actually made out of materials. Thus a surface must be thought of as having no thickness whatever. A curve or a straight line must be thought of as having length but no breadth or thickness. A point has only a position.

While we cannot manufacture such things as these, every one recognizes that they can be thought of in this ideal way What we can do is to represent them very nearly: thus a point is represented by a dot, a curve by a line drawn with pencil or chalk, a surface by paper or tin or some other very thin substance.

12. Angles. Two portions of straight lines that end at the same point form an **angle**. The two lines are called the **sides** of the angle, and the common end-point is called the **vertex** of the angle. The length of the sides has nothing to do with the size of the angle, which depends only on the amount of the opening.

Strictly speaking, when two portions of straight lines thus end at a common point, there are two angles formed: thus in the figure, the portions of straight lines BA and BC may be thought of as forming the angle marked x, or they may be thought of as forming the very much larger angle marked y which is all that is left of the plane if the angle marked x is taken away.

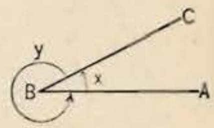


FIG. 9

We shall always understand that the smaller angle x is intended in such a case, unless the contrary is stated.

An angle may be read in any one of three ways:

(1) By the single letter at the vertex; as the angle B.

(2) By three letters, one on each side and one at the vertex; as, the angle ABC. The middle letter, here B, is always the one that stands at the *vertex* of the angle.

(3) By a single letter placed in the opening of the angle; as, the angle x.

13. Measurement of Angles. Units. The most usual *unit of angle* is the **degree** (°). It is formed as follows: Divide the circumference of any circle into 360 equal parts (or arcs), and then join the ends of one of these arcs to the center of the circle by straight lines; the angle thus formed at the center is *one degree.* Thus one degree is one three hundred sixtieth of one complete revolution.

The degree is subdivided into 60 equal parts, called ***minutes,*** (′). The minute is divided into 60 equal parts, called ***seconds*** (″). Thus an angle of 10 degrees, 20 minutes, and 15 seconds is written 10° 20′ 15″.

14. The Protractor. A **protractor** is an instrument for measuring angles. It is a half-circle made of cardboard, celluloid, or metal, with the center marked at O (Fig. 10) and with the circum-

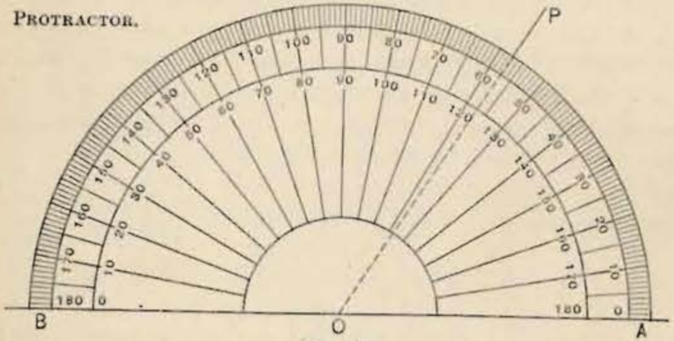


FIG. 10

ference divided by fine lines into 180 equal arcs. Each of these arcs corresponds to 1°, if the vertex of the angle is placed at O.

To measure an angle, place the protractor upon it so that one side of the angle lies along the radius OA, with the vertex of the angle at O. Then the other side of the angle will fall in some such position as OP, and the number of degrees and fractions of a degree can be read off directly from the scale

EXERCISES

1. Through how many degrees does the minute hand of a clock turn in fifteen minutes?

2. Through how many degrees does the hour hand of a clock turn in one hour?

3. The second hand of a watch turns on a circular dial that is divided into sixty equal parts. What is the angle between two successive marks? What is the angle between the mark for 10 seconds and the mark for 15 seconds? What is the angle between the mark for 10 seconds and that for 20 seconds?

4. Ordinary scales for weighing small objects are often made with a circular face like a clock face; the divisions of the scale indicate pounds; if the entire face represents 24 pounds, what is the angle between two successive pound marks?

5. There being 16 ounces in one pound, what is the angle between two successive ounce marks on the scale of Ex. 4?

6. How long does it take the minute hand of a clock to turn through 36°? How long does it take the hour hand of a clock to turn through 36°? 60°? 75°?

7. What weight will cause the hand of the scale described in Ex. 4 to turn through 15°? 60°? 75°? 150°?

8. Through how many degrees does a screwdriver turn in half a revolution?

9. If a wheel makes ten revolutions per minute, through how many degrees does it turn in one second?

10. Draw an angle of 150° with a protractor, and divide it into four equal parts by means of ruler and compasses. Measure the resulting angles with the protractor. How much error did you make in each case?

11. Draw an angle as nearly equal to 75° as you can judge by your eye. Then measure your angle with a protractor. How much error did you make? What fraction of 75° is your error?

15. Generation of Curves and Surfaces by Motion. We may think of curves and surfaces as formed, or *generated*, by motion.

If a point moves, its path is a curve.

If a curve moves, it generates a surface.

If a surface moves, it generates a solid.

Thus, the point of the compasses that draws a circle may be thought of as a moving point that is generating the circle. Again, if a circle is rotated about a line through its center, it generates the surface of a sphere. If a square that lies horizontally is lifted vertically, it generates a rectangular block; the block becomes a cube when the height through which the square is lifted becomes equal to one of its sides.

Notice, however, that a moving curve does not *always* generate a surface. Thus, when a wheel turns on its axle, the curve formed by its circumference is moving, but no surface is being generated. Likewise, a surface that merely slides upon itself does not generate a solid. All such motions as these are exceptions to the general rules stated above.

16. Generation of Angles by Rotation. An angle may always be thought of as generated by a line which rotates about one of its extremities (regarded as fixed). Thus, if the line AB rotates about the point A, it takes one after another the positions AC, AD, AE, AF, AG, and finally comes back to its original position AB. In each case it makes an angle with its original position AB, and it is to be noted that this angle increases as the rotation goes on.

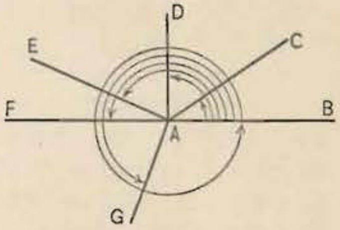


FIG. 11

17. Important Special Angles. Another important unit angle is the **right angle**, which is the angle between two lines that are **perpendicular** to each other (see footnote, p. 4).

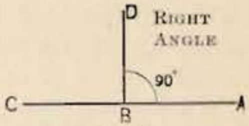


FIG. 12

Still another important angle is a **complete revolution**, the angle formed when a line turns around one of its extremities until it comes to its original position. A complete revolution is also called simply a **revolution**; or sometimes a *perigon*.

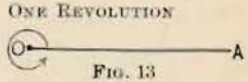


FIG. 13

When the two sides of an angle lie along the same straight line, and in opposite directions from the vertex, the angle is called a **straight angle**.

A straight angle is equal to two right angles, since a perpendicular to a straight line makes two equal right angles on one side of the line to which it is perpendicular.

A revolution is equal to two straight angles, or to four right angles.

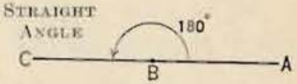


FIG. 14

Since a revolution is 360°, a straight angle is 180°, and a right angle is 90°.

An **acute** angle is an angle less than a right angle. An **obtuse** angle is an angle greater than a right angle and less than a straight angle.

ACUTE ANGLE OBTUSE ANGLE

FIG. 15

If two straight lines cross each other the sum of the two angles formed on the same side of either of the lines is a straight angle. If one of them is acute, the other is obtuse.

EXERCISES

1. In Fig. 11, pick out a right angle; an acute angle; an obtuse angle; an angle of one revolution; a straight angle. Are there any other right angles in the figure?

2. Show from the footnote on p. 4, regarding a perpendicular, that if two lines cross at right angles, any two of the four angles formed are equal to each other.

3. How many degrees are there in half a right angle? in one third of a right angle? in one fourth of a right angle?

4. How many degrees are there in one and one half right angles? in two right angles? What is another name for two right angles?

5. How many degrees are there in one revolution? in one fourth of a revolution? in one twelfth of a revolution? in one fifteenth of a revolution?

6. Is one fifth of a revolution an acute or an obtuse angle?

7. Is two thirds of a straight angle acute or obtuse?

8. Name two streets that you know which meet each other at right angles. Name two that do not. In the latter case describe the corner or corners at which there is an acute angle; those at which there is an obtuse angle.

9. Does a rafter of the roof of a barn make an acute or an obtuse angle with an upright in the side wall? What can you say of the angle at the peak of the roof where the rafters join?

10. Through what kind of an angle has a door turned on its hinges when it is said to be ajar? Can a door be opened through an obtuse angle?

11. The earth turns on its axis once in 24 hours. How many degrees of longitude correspond to 1 hour?

12. Apply the construction for dividing an angle into two equal parts (§ 9) to a straight angle. Show that the construction that results is the same as that of § 6.

18. Relations between Two Angles. Two angles that have a common vertex and one common side between them are called **adjacent** angles. Thus the angles a and b in the figure are adjacent angles.

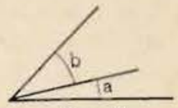


FIG. 16

If the sum of two angles is equal to a right angle, the two angles are said to be **complementary** to each other; or, either of them is called the **complement** of the other. Thus the angles x and y in Fig. 17 are complementary to each other.

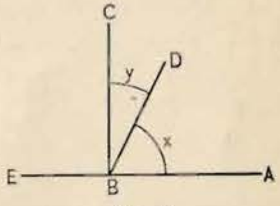


FIG. 17

If the sum of two angles is equal to two right angles, the two angles are said to be **supplementary** to each other; or, either of them is called the **supplement** of the other. Thus the angles x and y in Fig. 18 are supplementary to each other.

By the **sum** of two angles is meant the angle formed by placing the angles adjacent to each other; the sum is the total angle thus formed, as in the preceding figures. The measure of the sum in degrees is the sum of the number of degrees in the two given angles.

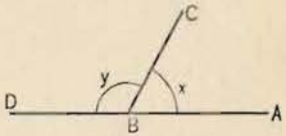


FIG. 18

19. Vertical Angles. If two lines AB and CD cross each other at a point O, the angles that lie opposite each other across the common vertex are called **vertical** angles.

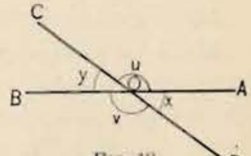


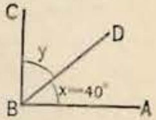
FIG. 19

Thus, in Fig. 19, the angles x and y are vertical angles; and u and v also are vertical angles. Since the sum of u and x is a straight angle, and the sum of u and y is a straight angle, it is easy to see that x and y must be equal. That is, *any two vertical angles are equal to each other.*

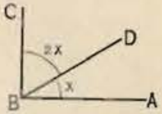
EXERCISES

[In this list, and hereafter, the sign ∠ is used for the word *angle*.]

1. Angle ABC is a right angle. If $\angle x = 40°$, how many degrees in $\angle y$? What is the complement of $40°$?



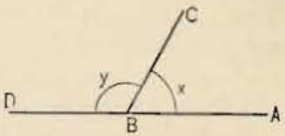
2. What is the complement of $30°$? $50° 20'$? $45° 18' 20''$? $74° 31' 14''$?



3. The complement of a certain angle x is $2x$. How many degrees are there in x?

4. The complement of a certain angle is eight times itself. What is the angle? Draw a diagram by means of a protractor.

5. In the figure, $\angle ABC + \angle CBD = 2$ right angles, or the straight angle ABD. If $\angle x = 50°$, how many degrees are there in $\angle y$? What is the supplement of $50°$?



6. What is the supplement of $80°$? $40° 15'$? $100° 30' 20''$?

7. The supplement of a certain angle x is $4x$. How many degrees are there in x? Draw a diagram.

8. The supplement of a certain angle is eleven times itself. What is the angle? Draw a diagram.

9. Compare the complements of two equal angles.

10. Comparė the supplements of two equal angles.

11. What kind of an angle is equal to its supplement?

12. Find two complementary angles whose difference is $36°$.

13. Two supplementary angles are such that one is $40°$ more than the other. Find each of the angles.

14. One line meets another line so that one angle is five times its adjacent angle; find each of the angles.

15. How do two angles u and v compare if they have the same complement? if they have the same supplement?

20. Contrast between Drawing and Construction of Figures. One purpose of a part of our study will be to show how figures can be drawn *without any other instruments than a ruler and compasses;* for distinctness, we shall say that a figure has been **constructed** when only these instruments have been used. The direction to *construct* a figure will carry with it the direction that only these instruments are to be used.

There is no objection whatever to the use of other instruments for quickly sketching a figure. Thus perpendiculars may be drawn by means of a fixed square, such as that used by carpenters or draftsmen. We shall continue to use the words "*to draw a figure*" whenever we intend that other drawing instruments than ruler and compasses may be used.

When a figure is to be drawn only in a very rough fashion, for example by free-hand without any instruments, we shall say that it is to be *sketched*.

21. Reduction or Enlargement of Drawings. It is often inconvenient or impossible to draw a figure its actual size, or, as is often said, *life-sized*. Thus a plan of a house cannot conveniently be drawn the size of the house.

In such cases, a figure is drawn in which every distance is reduced in the same ratio. Thus in a figure drawn half size, the distances in the figure are all half the actual distances.

House plans are usually drawn on a scale which makes a distance of one quarter of an inch on the drawing represent one foot in the actual house; that is, the scale is reduced in the ratio of one to forty-eight. The angles, of course, remain unchanged.

An accurate record of the scale used should be written on the face of every drawing that is not life-size. Many geometric figures do not change their properties when merely enlarged, and for the purpose of showing something about a figure, it may be drawn in any convenient size. Thus angles are not changed by reducing the size of the drawing.

Figures described in exercises in which small distances are used should be enlarged when they are drawn on the blackboard. When large distances are mentioned in exercises, the size may be reduced for a drawing on paper or at the board.

EXERCISES

1. What distance on the drawing represents 20 ft. in a house plan drawn to the scale mentioned in § 21 ($\frac{1}{4}$ in. to 1 ft.)?

2. What actual distance does 6 in. represent in a house plan drawn to the scale mentioned in § 21?

3. What are the real dimensions of a room that appears on a house plan to be $2\frac{1}{2}$ by 3 in., if the plan is drawn on the scale described in § 21? What is the actual floor area of the room? (The area is the length times the breadth.)

4. A table 2 ft. 6 in. wide and 4 ft. long is to be placed in one of the rooms. How large a spot will it represent in the drawing on the scale of § 21?

5. On a map whose scale is 37 mi. to the inch, the distance between Chicago and Ann Arbor is $5\frac{5}{8}$ in. What is the actual distance?

6. New York is 143 mi. from Albany. How far apart are they on the map referred to in Ex. 5?

7. A ship on leaving port sails N.W. 18 mi., then N. 15 miles. Draw a map showing her course, using a scale of 1 in. for 10 mi. In this manner find (by measurement on your map) her approximate distance and her bearing from port; that is, how many degrees West of North.

8. When a vertical pole 20 ft. high casts a shadow 35 ft. long, what is the acute angle the sun's rays make with the horizontal?

[This angle is called the **angle of elevation** of the sun. Draw a map, scale 10 ft. to the inch, and use a protractor to measure the angle.]

22. Triangles. Notation. A figure bounded by three straight lines is called a **triangle**. The bounding lines are called the *sides*, and the points where the sides meet are called the *vertices*. Usually, the small letters, a, b, c, are used to denote the sides, while the capital letters, A, B, C, are used to denote the vertices. The side a is then always placed opposite the vertex A, while b is likewise placed opposite to B, and c opposite to C, as indicated in the figure.

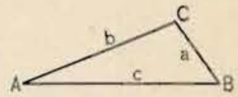


FIG. 20

The angle at A is called the **included angle** of the sides b, c. Similarly, B is the included angle of the sides a, c; and C is the included angle of a, b.

The angles at A, B, C are known as the **interior angles** of the triangle. Besides these, every triangle has what are known as **exterior angles**. In the figure, x represents the exterior angle at B. In general, an **exterior angle** is one which, like x, is formed between one side of the triangle and the prolongation (extension) of another side.

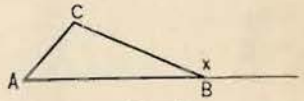


FIG. 21

23. Circles. A **circle** is a closed curve, every point of which is equally distant from a fixed point within called the **center**. The distance from the center to any point on the circle is called the **radius**. A straight line through the center terminated by the circle, is called a **diameter**.

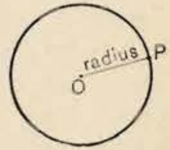


FIG. 22

It follows from this definition of a circle that all its radii are equal. How do the radius and the diameter compare in length?

When several circles have the same center, but different radii, they are called **concentric**. Draw three concentric circles.

The curve that forms the circle is often called the **circumference**. The word *circle* is sometimes used to denote the *space enclosed* by the circumference.

24. Squares and Rectangles. Any figure bounded by four lines is called a **quadrilateral**.

A **rectangle** is a quadrilateral each of whose angles is a right angle. The two sides that meet at any corner (vertex) of a rectangle are therefore perpendicular to each other.

A line that joins opposite corners (vertices) of a rectangle is called a **diagonal**. Thus AC in Fig. 23 is a diagonal of the rectangle $ABCD$.

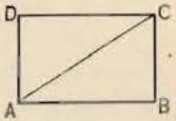


Fig. 23

If all the sides of a rectangle are of equal length, it is called a **square**.

25. Areas. To measure an area of any sort, a unit is usually chosen which is a square, any side of which is equal to the unit of length. The most common unit of area is one square foot; that is, a square each of whose sides is 1 ft. long.

Any given area is measured by comparing its size with that of the unit square. In particular, the *area of any rectangle* is found to be the product of the number of units of length in its base times the number of units of length in its height. This rule is usually learned in Arithmetic.

Areas that are bounded by curved lines or by pieces of straight lines are usually measured by supposing them filled up with little squares, each of whose areas we can find. If the area bounded by the figure cannot be precisely filled in this way, at least it is greater than the sum of the areas of those squares that lie entirely within it; and it is less than the sum of the areas of squares that entirely cover it.

A good practical way to estimate the area of any figure is to draw it on paper that is ruled into little squares of known size. Such paper (called squared paper, or cross section paper) can usually be bought at any stationers, the ruling being into squares one tenth of an inch on each side. There are, of course, one hundred such squares in one square inch.

Beneath, in Fig. 24, several figures are drawn on a sheet of squared paper. Estimate the areas of each of them in the manner just described.

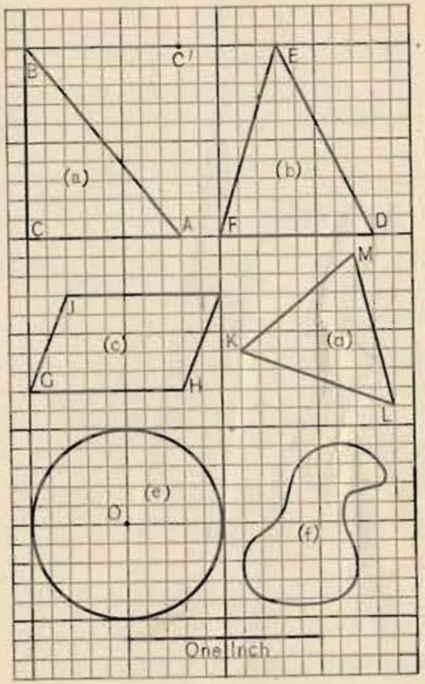


Fig. 24

The very best conception of area is that which results by imagining the rulings on the squared paper just mentioned to be made finer and finer, so that the estimated area becomes more and more nearly the correct area. The ideal or exact area bears the same relation to these estimates that the drawings made by human beings do to the ideal figures in geometry. (See § 11.)

In the same way, lengths of curved lines can be estimated by first replacing each small bit (arc) of the curved line by a straight line joining the ends of the arc, and then taking the sum of the lengths of all these pieces of straight lines. The smaller the arcs, the more accurate this result.

EXERCISES

1. How many exterior angles has a triangle?

2. In a certain triangle ABC the interior angle at A is 49°. What is the exterior angle at the same vertex?

3. Draw three triangles of different shapes, and then, using the protractor, determine the sum of the three interior angles for each triangle. Are the three sums equal, and if so, to what?

4. The end of the minute hand of a clock always travels in a circle. Why?

5. Draw on a piece of squared paper a rectangle $\frac{1}{2}$ in. wide by 1 in. high. Draw its diagonal. Estimate the area in each triangle into which the rectangle is divided. Are the areas of the two triangles equal?

6. Draw on squared paper a triangle with one right angle, and with the perpendicular sides 1.5 in. and .8 in. long, respectively. Estimate its area.

7. Draw on squared paper a rectangle whose diagonal is the longest side of the triangle mentioned in Ex. 6. Find its area.

8. Draw on squared paper two concentric circles, with the radius of one twice that of the other. Estimate their areas.

9. *Construct*, on squared paper, a triangle whose sides are, respectively, 2 in., 1.5 in., and 1.7 in. Estimate its area.

[The area can also be found by dividing the triangle into two triangles that have one right angle in each, by a perpendicular from one corner to the opposite side, and then completing each of these smaller triangles into rectangles, as in Exs. 5 and 7.]

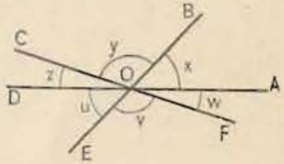
10. What is the sum of the four angles of a rectangle?

11. A courtyard is 25 yd. long by 15 yd. wide. Draw a plan of it on squared paper, scale 10 yd. to the inch. What area is represented by one of the ruled squares on your paper? Find the area of the courtyard.

MISCELLANEOUS EXERCISES FOR PART II

1. In Fig. 19, p. 17, the angles x and y are vertical angles. If $\angle x = 40°$, what is the value of $\angle u$, its supplement? Since $\angle u$ is also the supplement of $\angle y$, what is the value of $\angle y$? Compare $\angle x$ and $\angle y$.

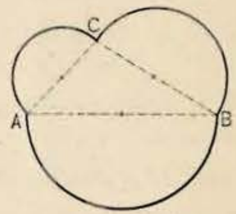
2. If $\angle x$ in Fig. 19 is 38°, what is the value of y? If $\angle y$ is 78°, what is the value of $\angle x$?



3. Name pairs of vertical angles in the adjoining figure. What is the value of $\angle x + \angle y + \angle z$?

4. Draw a diagram showing the complement and the supplement of an acute angle ABC. What is their difference?

5. Show that the bisectors of two adjacent supplementary angles are perpendicular to each other.



6. By use of squared paper determine (approximately) the number of square inches in the adjoining figure, taking $AB = 4$ in., $BC = 3$ in., and $CA = 2$ in.

7. The figures (*a*), (*b*), (*c*) of Fig. 24, p. 23, have the same area. Is the length of the boundary therefore the same for each of them? Estimate these lengths.

8. In what ratio is the drawing of a house reduced from the actual house if a distance of 12 ft. is represented by a line $1\frac{1}{2}$ in. long? What actual distance is represented on the same drawing by a line 2 in. long?

9. Find the difference in longitude at two places on the earth if the difference in sun time is 2 hr. 30 min.

10. Find the difference in the sun time between two places that differ in longitude by 30°; between two places that differ in longitude by 20°.

PART III. STATEMENTS FOR REFERENCE

26. Assumptions. In Chapters I–V, which we are about to study and in which many of the principles thus far used are more carefully considered, we shall make use of certain self-evident *general* statements. These statements are those upon which Geometry is based. They are divided into two classes, known respectively as **Axioms** and **Postulates**. *Axioms* refer to quantities in general, that is, without special regard to geometry; *postulates* refer especially to geometry. The following lists (§§ 27, 28) contain axioms and postulates that will be clear at this time.

27. Axioms.

1. *If equals are added to equals, the sums are equal.* Thus, if $a = b$ and $c = d$, then $a + c = b + d$.

2. *If equals are subtracted from equals, the remainders are equal.* Thus, if $a = b$ and $c = d$, then $a - c = b - d$.

3. *If equals are multiplied by equals, the products are equal.* Thus, if $a = b$ and $c = d$, then $ac = bd$.

4. *If equals are divided by equals, the quotients are equal.* Thus, if $a = b$ and $c = d$, then $\frac{a}{c} = \frac{b}{d}$. In applying this axiom it is supposed that c and d are *not equal to zero.*

5. *If equals are added to unequals, the results are unequal and in the same order.* Thus, if $a = b$ and $c > d$, then $a + c > b + d$.

6. *If equals are subtracted from unequals, the results are unequal and in the same order.* Thus, if $a > b$ and $c = d$, then $a - c > b - d$.

7. *If unequals are added to unequals in the same sense, the results are unequal in the same order.* Thus, if $a > b$ and $c > d$, then $a + c > b + d$.

8. *If unequals are subtracted from equals, the results are unequal in the opposite order.* Thus, if $a = b$ and $c > d$, then $a - c < b - d$.

9. *Quantities equal to the same quantity, or to equal quantities, are equal to each other.* In other words, *a quantity may be* **substituted** *for its equal at any time in any expression.*

10. *The whole of a quantity is greater than any one of its parts.*

11. *The whole of a quantity is equal to the sum of its parts.*

28. Postulates.

1. *Only one straight line can be drawn joining two given points.*

2. *A straight line can be extended indefinitely.*

3. *A straight line is the shortest curve that can be drawn between two points.*

4. *A circle can be described about any point as a center and with a radius of any length.*

5. *A figure can be moved unaltered to a new position.*

6. *All straight angles are equal.* Hence, also, *all right angles are equal*, for a right angle is half of a straight angle. § 17.

29. Names of Statements. Aside from the above axioms and postulates, the words *Theorem*, *Problem*, *Proposition*, and *Corollary* will hereafter be used in the following special senses:

Theorem. A statement of a fact which is to be, or has been, proved is called a theorem.

Problem. A statement of a construction (see § 20) which is to be made is called a problem.

Proposition. Either a theorem or a problem is known as a proposition.

Corollary. A theorem which follows immediately as a consequence of some other theorem is called a corollary of it.

30. Summary of Construction Problems. In this Introduction we have shown how to make the following constructions (with ruler and compasses alone):

1. To construct a triangle, each of whose sides is equal to a given length. § 3, p. 2.

2. To construct a triangle, whose three sides are, respectively, equal to three given lengths. § 4, p. 3.

3. To construct a perpendicular to a given straight line at a given point in that line. § 5, p. 4.

4. To construct a perpendicular to a given line from a given point not on that line. § 6, p. 5.

5. To construct, at a given point in a given line, another line that makes an angle equal to a given angle with the given line. § 7, p. 6.

6. To divide a portion of a straight line into two equal parts. (To bisect a line.) § 8, p. 7.

7. To divide a given angle into two equal parts. (To bisect an angle.) § 9, p. 8.

31. Facts or Theorems now Known. We have also either assumed or proved the following geometrical facts:

1. All radii of the same circle are equal. § 2, p. 1; and § 23, p. 21.

2. Circles whose radii are equal can be placed upon each other so that their centers and their circumferences coincide (lie exactly upon each other).

3. Equal angles may be placed upon each other so that their vertices coincide and their corresponding sides fall along the same straight lines. This is, in fact, what we mean by *equal* angles.

4. Two straight lines have at most one point in common. See postulate 1, p. 27.

5. Two circles have at most two points in common. See § 8.

6. A straight line and a circle may have at most two points in common.

7. At a given point in a given line only one perpendicular can be drawn to that line. (A consequence of Problem 3, § 5.)

8. Complements of the same angle, or of equal angles, are equal. Ex. 9, p. 18.

9. Supplements of the same angle, or of equal angles, are equal. Ex. 10, p. 18.

10. Vertical angles are equal. § 19.

11. If two adjacent angles have their exterior sides in a straight line, they are supplementary. § 18.

12. If two adjacent angles are supplementary, they have their exterior sides in a straight line. See Fig. 18.

13. If each of two figures can be placed upon a third figure so as to coincide with it, they can be placed upon each other so that they coincide.

14. Any desired angle may be drawn, and any angle may be measured, by the use of a protractor. (But the use of this instrument is not permitted when a figure is to be *constructed*. See § 20.)

15. A perpendicular to a given line through any given point may be drawn by means of a set square or a drawing triangle. (But the use of these instruments is not permitted when a figure is to be *constructed*. See § 20.)

16. The area of a rectangle (in terms of a unit square) is equal to the product of its width and its height, measured in units of length equal to one side of the unit square.

17. The area of any given figure is greater than the area of any figure that is drawn completely within it.

18. The areas of two figures are equal if they consist of corresponding portions that can be made to coincide.

SUPPLEMENTARY EXERCISES

1. Draw (using protractor) an angle of 50°. Construct (using ruler and compasses) its complement. Measure your new angle with the protractor and see if it has the proper number of degrees.

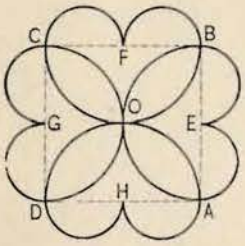
2. Draw an acute angle and then an obtuse angle. In each case estimate as nearly as you can without using the protractor how many degrees there are in the angle. Then, check your estimate by measurement, note your errors, and find what fraction of the correct amounts each of these errors is.

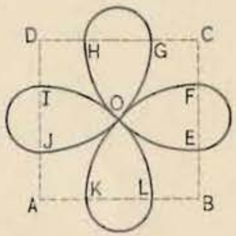
3. Find the angle whose complement and supplement are in the ratio 4 : 13.

4. Show how to construct an angle of 45°; an angle of 22° 30′.

5. Construct two lines that bisect (divide into two equal parts) each other at right angles.

6. Draw the patterns shown below. Your drawings should be twice the size of the copies. The curves are formed by joining arcs of certain circles.

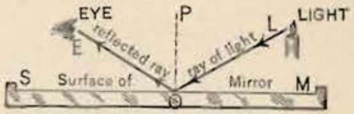




7. A traveler wishes to go due north but finds his way barred by a swamp. He therefore goes five miles northeast, then five miles north, then five miles northwest, and he now finds himself due North of his starting point. Draw a map (scale one mile to the inch) and determine by measurement how many miles he lost by going out of his course.

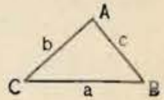
8. A tower is observed from a point 500 ft. distant from its foot, and the angle the line of sight makes with the horizontal is found to be 15°. What is the height of the tower?

9. It is a principle of physics that when a ray of light is reflected from a mirror, the reflected ray makes the same angle with a line perpendicular to the mirror that the original ray makes with the same line. Show that the original ray and the reflected ray also make equal angles with the mirror itself.

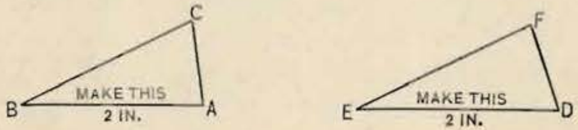


10. Two forts defending the mouth of a river, one on each side, are 10 mi. apart. Their guns have a range (possible shooting distance) of $4\frac{1}{2}$ mi. Draw a plan (scale 1 mile to the inch) showing what part of the river is exposed to fire from the two forts. [GODFREY AND SIDDONS.]

11. Construct two triangles each with the sides a, b, and c, as indicated in the adjacent figure. See § 4, p. 3. Cut them out and place one on the other so that they coincide. What conclusions do you draw concerning such triangles?



12. Construct two triangles each with the base $AB = 2$ in., angle $ABC =$ angle DEF, and angle $ACB =$ angle DFE. Cut one of these triangles from the paper and place it upon the other so that the corresponding parts coincide. What conclusions do you draw concerning such triangles?



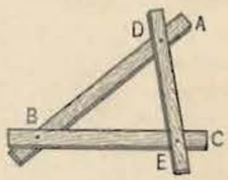
13. The length of a rectangular field is 50 yd. and the length of the entire boundary is 180 yd. What is its breadth and its area?

14. Construct two equal angles, ABC and DEF. On the sides of these angles lay off the distances BA and ED, each 2 in.; also the distances BC and EF, each $1\frac{1}{2}$ in. Join

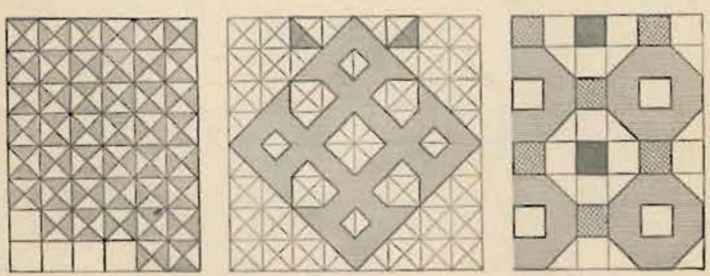
AC and DF, thus forming two triangles ABC and DEF. What conclusions do you draw concerning such triangles?

15. Draw a triangle with two sides $AC = 3$ in. and $AB = 5$ in. and their included angle $= 35°$. (Use protractor.) Draw another triangle in which $AC = 3$ in., $AB = 5$ in., but their included angle $= 20°$. Do two sides alone fix (determine) a triangle?

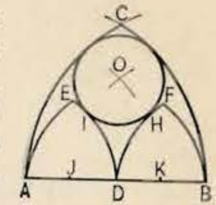
16. Is a triangle determined if two sides and their included angle are given? Show the relation of the last question to the following fact from everyday life: Two pieces of board AB and BC hinged at B can be held rigid by nailing a crosspiece DE to both sides.



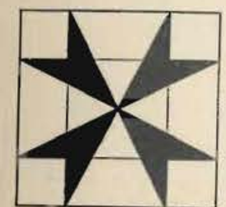
17. Many designs may be made by emphasizing a part of the lines on squared paper, or the diagonals of those squares. Copy and complete the following; and also invent others.



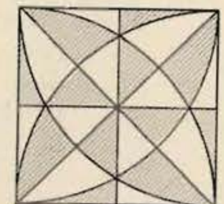
18. To construct the plan for a Gothic window, proceed as follows: Take any line AB and divide it into four equal parts. With A and B as centers and a radius equal to AB, draw arcs intersecting at C. What radius and what centers are used to describe the small arcs? O is found by taking A and B as centers and AK as a radius. Complete the figure.



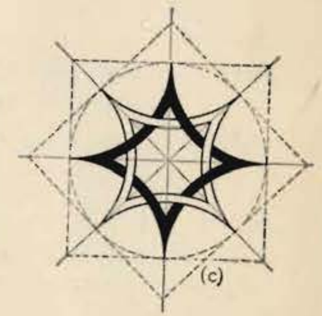
19. Construct (by ruler and compasses) patterns like the following:



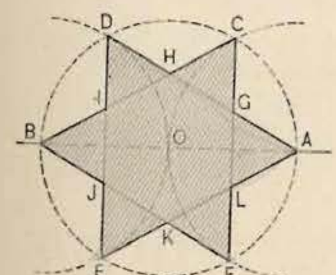
(a)



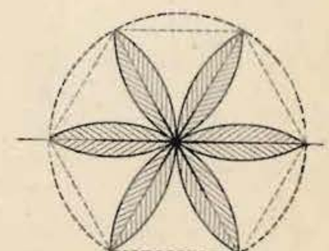
(b)



(c)



(d)



(e)

SYMBOLS AND ABBREVIATIONS

The following symbols and abbreviations will be used for the sake of brevity throughout the present book:

$=$	equal, or is equal to
$\neq$	not equal, or is not equal to
$>$	greater than
$<$	less than
$\cong$	is congruent to
$\perp$	perpendicular, or is perpendicular to
$\parallel$	parallel, or is parallel to
$\sim$	similar, or is similar to
$\angle$	angle
$\measuredangle$	angles
$\triangle$	triangle
$\triangle$	triangles
$\square$	parallelogram
$\square$	parallelograms
$\bigcirc$	circle
Ⓢ	circles
$\frown$	arc

Ax.	Axiom
Cons.	Construction, or by construction
Cor.	Corollary
Def.	Definition
Hyp.	Hypothesis, or by hypothesis
Iden.	being identical
Prop.	Proposition
rt.	right
st.	straight
Th.	Theorem
Prob.	Problem
Fig.	Figure or diagram
. . . .	*and so on*
$\therefore$	*hence* or *therefore*

The signs $+$, $-$, $\times$, $\div$, are used with the same meanings as in algebra. The following agreements are also made:

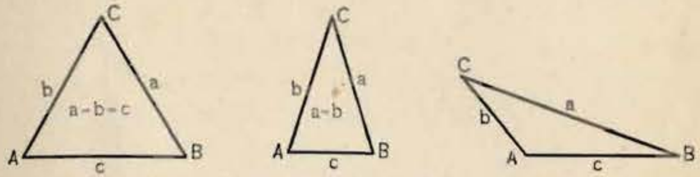
$$a \times b = a \cdot b = ab$$
$$a \div b = a/b = a:b$$

CHAPTER I

RECTILINEAR FIGURES

PART I. TRIANGLES

32. Definitions. It is desirable to distinguish between several kinds of triangles as follows:



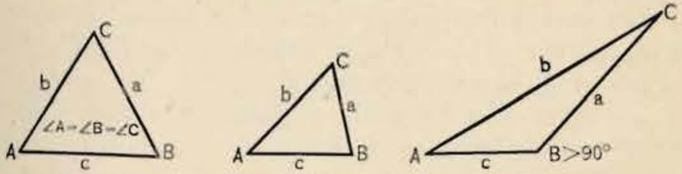
EQUILATERAL TRIANGLE ISOSCELES TRIANGLE SCALENE TRIANGLE

FIG. 25

An **equilateral** triangle has all three of its sides equal.

A triangle that has any two of its sides equal to each other is called an **isosceles** triangle.

A **scalene** triangle is one that has no two of its sides equal.



EQUIANGULAR TRIANGLE ACUTE TRIANGLE OBTUSE TRIANGLE

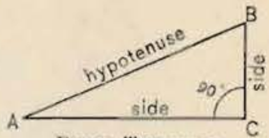
FIG. 26

An **equiangular** triangle has all three of its angles equal.

An **acute** triangle is one whose angles are all acute.

An **obtuse** triangle is a triangle that has one obtuse angle.

A triangle one of whose angles is a right angle is called a **right triangle**. In such a triangle, the side opposite the right angle is called the **hypotenuse**, and the word "side" is used only for the other two sides, as indicated in Fig. 27.



RIGHT TRIANGLE

FIG. 27

The side upon which any triangle appears to rest is called its **base**.

The vertex opposite to the base is called the **vertex** of the triangle, and the angle opposite to the base is called the **angle at the vertex**.

The perpendicular distance from any vertex to the opposite side (extended if necessary) is called an **altitude** of the triangle.

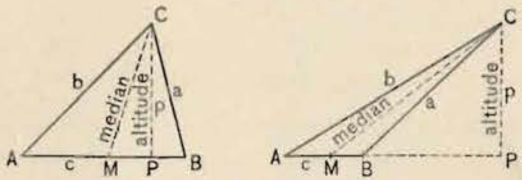


FIG. 28

The distance from any vertex to the middle point of the opposite side is called a **median** of the triangle.

Any portion of a straight line between two points is called a **segment** of that line. Thus the sides, altitudes, and medians of any triangle are **line segments**.

Any figure composed wholly of points and straight lines is called a **rectilinear** figure.

33. Congruent Figures. Two triangles that can be made to fit each other exactly (coincide) by properly placing the one upon the other are called **congruent**. More generally, any two geometric figures are **congruent** if they can be made to coincide exactly.

When two congruent figures are made to coincide, any corresponding parts coincide: *corresponding angles of congruent figures are equal; corresponding lengths are equal;* any portion of one of two congruent figures is congruent to the corresponding portion of the other.

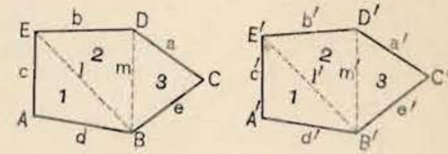


FIG. 29

An illustration of two congruent figures, each broken up into certain corresponding parts which are also congruent, is given in Fig. 29.

EXERCISES

1. How many altitudes has a triangle? How many medians? Draw figures illustrating your answers.

2. Can a right triangle be isosceles? Draw figures.

3. Are all right triangles isosceles? Draw figures.

4. Can a right triangle be equilateral? Explain your answer.

5. What kind of triangle was drawn in Problem 1, p. 2? In Problem 2, p. 3?

6. Investigate the following questions: (1) Is there any kind of triangle for which the medians coincide with the altitudes, and also with the bisectors of the three angles? If so, describe it. (2) Will these lines usually all be different for a triangle? Illustrate your answer by drawings.

34. Congruence of Triangles. We now proceed to state and prove certain theorems regarding triangles. The pupil should first reread carefully § 22.

Our first topic of study will be the following question: "When are two triangles congruent?" Note again the definition of congruent figures as given in § 33.

35. Theorem I. ***If two triangles have two sides and the included angle of the one equal, respectively, to two sides and the included angle of the other, the triangles are congruent.***

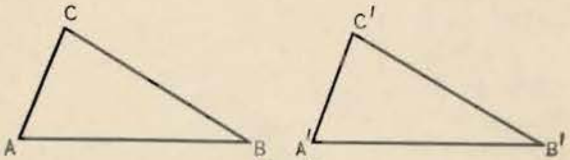


FIG. 30

In the figure, let ABC and $A'B'C'$ be the two triangles, and let us suppose that we know (as the theorem says) that $AB = A'B'$, $AC = A'C'$, and that the angle $A =$ the angle A'.

We are now to prove that these two triangles are congruent, which means that we have to show that the one may be fitted on to the other so that they will exactly coincide in all their parts.

Now, since $AB = A'B'$, we can place the triangle $A'B'C'$ on the triangle ABC so that $A'B'$ will coincide with its equal AB, making the point C' fall somewhere on the same side of AB as C.

Then, $A'C'$ will lie along AC, because angle $A' =$ angle A, this being also one of the given (supposed) facts.

Moreover, C' will fall exactly at C, since $A'C' = AC$, which is another of the given facts.

It thus follows that the side $C'B'$ will fit exactly upon CB, for, according to Postulate I, only one straight line can be drawn through the two points, C, B.

Therefore, the two triangles are congruent.

NOTE. The fact stated in this theorem was indicated in Exs. 2 and 3, p. 6; and in Ex. 14, p. 32. In those exercises, however, the truth was only suggested. What we do in a *proof* such as that just given is to make certain what was previously simply plausible.

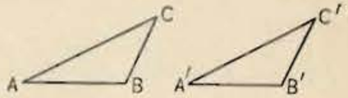
36. Corollary I. ***Two right triangles are congruent if the two sides of the one are equal respectively to the two sides of the other.***

Reasoning from Theorem I, the student should convince himself of the truth of this corollary.

[HINT. First, note the form of all right triangles, as illustrated in Fig. 27. Note also from the same figure how the word "side" is used for such triangles. Now draw two such triangles having the "sides" of the one equal respectively to the "sides" of the other, and see how Theorem I applies to them.]

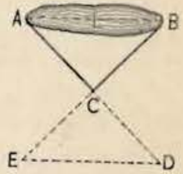
EXERCISES

1. Write out the proof of Theorem I for two triangles shaped as in the adjoining figure, in which we suppose $AB = A'B'$, $AC = A'C'$, and angle $A =$ angle A'.



2. Using a protractor, draw a triangle in which one angle is 30° and the two sides that include that angle are respectively 3 and 4 inches long. Show, by Theorem I, that if another triangle is drawn which has these parts, it is congruent to the one first drawn.

3. In the figure, L represents a lake. It is required to find its length, *i.e.* the distance between A and B. Show that this may be done as follows:



(1) Fix a stake at some convenient point C, and measure the distances AC and BC.

(2) In line with AC set a stake D, such that $CD = AC$. Then in line with BC, set a stake E, such that $CE = BC$.

The distance from stake D to stake E will be the required distance between A and B. Show that this is a consequence of Theorem I.

4. Show how the method of Ex. 3 could be applied to determine the greatest length of your school building.

37. Theorem II. *If two triangles have two angles and the included side in the one equal, respectively, to two angles and the included side in the other, the triangles are congruent.*

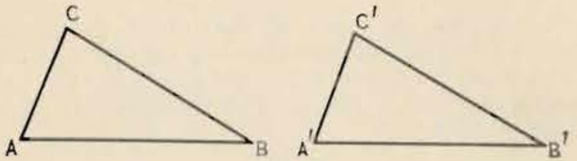


Fig. 31

In the figure, let ABC and $A'B'C'$ be the two triangles, and let us suppose that we know that

$$\angle A = \angle A', \angle B = \angle B', \text{ and } AB = A'B'.$$

[For brevity we here use the symbol $\angle$ for the word angle (see p. 34). Also, instead of "side AB = side $A'B'$" we write simply $AB = A'B'$.]

We shall now prove that the triangles ABC and $A'B'C'$ are congruent by showing that it is possible to place the one upon the other so that they shall coincide.

To carry out the proof, place the triangle $A'B'C'$ upon the triangle ABC so that $A'B'$ coincides with its equal AB, and so that C' and C fall upon the same side of AB.

Then $A'C'$ will fall along AC because $\angle A = \angle A'$, which was one of the given facts.

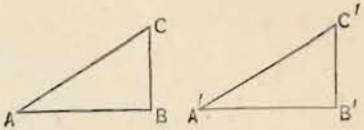
Moreover, $B'C'$ will fall along BC, since $\angle B = \angle B'$, which was also given.

It follows from this that C' will fall exactly at C, for, by Statement 4, § 31, the two lines AC and BC can intersect in only *one* point.

Thus, the two triangles can be made to coincide completely, and they are, therefore, congruent. Using the symbols of p. 34, this result is written in the following form:

$$\triangle ABC \cong \triangle A'B'C'.$$

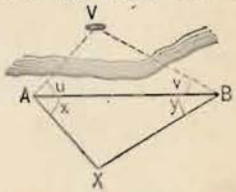
38. Corollary 1. *Two right triangles are congruent if an acute angle and its adjacent side in one are equal, respectively, to an acute angle and its adjacent side in the other.*



Thus, the two right triangles ABC and $A'B'C'$ are congruent if $AB = A'B'$ and $\angle A = \angle A'$. This corollary, like all others, should be proved by the student.

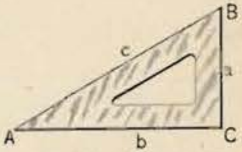
EXERCISES

1. Draw a right triangle having one angle 30° and the adjacent side (not the hypotenuse) 3 in. long. Measure the other angle with a protractor, and measure each of the other sides. Repeat this with an angle of 45° in place of 30°.



2. Theorem II was used by Thales (640–546 B.C.) to determine the distance of a vessel V from the shore, by measuring the angles u and v and then constructing the congruent triangle ABX on shore. Explain how this can be done.

3. A drawing triangle is usually made of celluloid, with one angle ($\angle C$ in the figure), a right angle. One other angle A is usually made 30°. Show that if the length AC is chosen, the form of the triangle is completely determined.



Drawing Triangle

39. Form of Proofs. Hereafter we shall use symbols whenever possible, thus enabling us to give proofs in a more condensed form. See the list of symbols, p. 34.

Note that in the arrangement of the proof the different steps are in the left column, while the reason for each step is in the right column.

40. Theorem III. *In an isosceles triangle the angles opposite the equal sides are equal.*

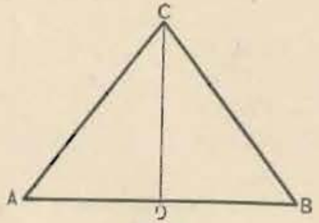


Fig. 32

Given the isosceles triangle ABC in which $AC = BC$.

To prove that $\angle A = \angle B$.

Proof. Draw CD bisecting $\angle ACB$.

Then, in the $\triangle ACD$ and BCD we have

	$CD = CD$,	Iden.
	$AC = BC$,	Given
	$\angle ACD = \angle BCD$.	Cons.
Therefore	$\triangle ACD \cong \triangle BCD$;	§ 35
and hence	$\angle A = \angle B$.	§ 33

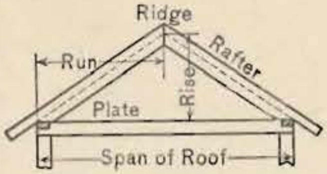
41. Corollary 1. *If a triangle is equilateral, it is also equiangular.*

State carefully the reasons for this conclusion, in the form of the proof of Theorem III.

EXERCISES

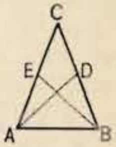
1. Construct (using ruler and compasses) an isosceles triangle and measure by means of the protractor the angles opposite the equal sides. Do your results conform to what Theorem III says?

2. An ordinary gable roof has a form (cross section) such as indicated in the accompanying figure. How does this illustrate Theorem III?



3. Mention some other familiar object in which an isosceles triangle occurs, draw a figure to represent it, and state how it illustrates Theorem III.

4. Prove that the medians drawn through the base angles of an isosceles triangle are equal. Write out the proof in the precise form used in the proof of Theorem III, using two columns, one for statements of fact and the other for the reasons; and use the symbols of p. 34.



[Hint. *Given* the isosceles $\triangle ABC$ in which $AC = BC$ and let AD and BE be the medians drawn through the base angles A and B. *To prove* $AD = BE$. In the proof, direct your attention to the $\triangle EAB$ and DBA. They have $\angle EAB = \angle DBA$ by Theorem III; at the same time they have $AE = BD$ because each is one half of the equal sides AC and BC. Also $AB = AB$. From these facts, reason by means of Theorem I that $\triangle ABE \cong \triangle ABD$, and hence $AD = EB$, which was to be proved.]

5. Prove that the bisectors of the base angles of an isosceles triangle are equal.

[Hint. Draw a figure similar to that of Ex. 4, but make AD bisect $\angle A$, and BE bisect $\angle B$. Direct your attention to the $\triangle EAB$ and DBA. They have $\angle EAB = \angle ABD$ by Theorem III; at the same time $\angle EBA = \angle DAB$, being halves of equal angles. Whence, show by Theorem II that $\triangle EAB \cong \triangle DBA$. Then, $AD = BE$, as was to be proved.]

42. Nature of Proofs in Geometry. Hypothesis. Conclusion. Every proof, as arranged in the form which we have now adopted and as illustrated by the proof of Theorem III, consists of four parts, as follows:

(1) A careful statement of the theorem, accompanied by an appropriate figure.

(2) A condensed statement of what we have to work with, or what is *given*. This part of the statement of the theorem is called its **hypothesis**, or thing supposed. Thus, in Theorem III the *hypothesis* is that the triangle is isosceles.

(3) A condensed statement of what is to be proved. This is called the **conclusion**.

Thus, in Theorem III the *conclusion* is that "the angles opposite the equal sides are equal."

(4) The details of the proof. This consists of a series of statements, each accompanied with its reason.

EXERCISES

1. What is the hypothesis of Theorem I? What is its conclusion? Answer the same questions for Theorem II.

2. Hypotheses and conclusions occur in all careful arguments in other subjects as well as in Geometry. Pick out the hypothesis and the conclusion in each of the following statements:

(*a*) If he is guilty, he should be punished.

(*b*) If a piece of iron is magnetized, it will attract other pieces of iron.

(*c*) A body heavier than water will sink in water.

(*d*) Corollaries of §§ 36, 38, 41.

3. Make some conditional statement similar to those of Ex. 2, and give the hypothesis and the conclusion.

4. Make some statement that you think is true about some geometric figure. Pick out the hypothesis and the conclusion. Draw a figure to illustrate the statement.

43. Theorem IV. *The bisector of the angle at the vertex of an isosceles triangle is perpendicular to the base and bisects the base.*

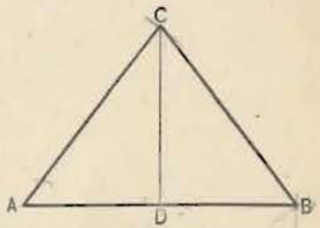


Fig. 33

Given the isosceles $\triangle ABC$ in which CD bisects the vertical $\angle C$.

To prove that $CD \perp AB$ and that $AD = DB$.

Proof. In the $\triangle s$ ADC and DBC we have

	$AC = CB$,	Given
and	$\angle ACD = \angle DCB$.	Given
Moreover	$CD = CD$.	Iden.
Therefore	$\triangle ADC \cong \triangle DBC$;	§ 35
whence	$\angle ADC = \angle BDC$,	
so that	$CD \perp AB$.	§ 17
Also	$AD = DB$.	§ 33

44. Corollary 1. *In any isosceles triangle* (*a*) *The bisector of the angle at the vertex divides the triangle into two congruent right triangles.*

(*b*) *The bisector of the vertical angle coincides with both the altitude and the median drawn through the vertex.*

(*c*) *The perpendicular bisector of the base passes through the vertex, and divides the triangle into two congruent right triangles.*

EXERCISES

1. If a plumb line (a string with a heavy bob attached at one end) be let down from the ridge of the roof represented in Ex. 2, p. 43, the bob strikes the floor in a way that illustrates Theorem IV. How?

2. Prove that if the bisector of the angle at the vertex of any triangle is perpendicular to the base, the triangle is isosceles.

[Hint. Apply Theorem II to the $\triangle s$ ADC, BDC (Fig. 33), and show that $AC = CB$.]

45. Theorem V. ***If two triangles have three sides of the one equal respectively to the three sides of the other, they are congruent.***

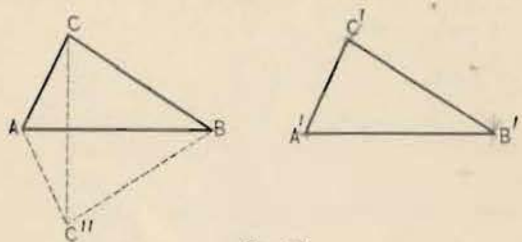


FIG. 34

Given the $\triangle ABC$ and $A'B'C'$ in which $AB = A'B'$, $BC = B'C'$, and $CA = C'A'$.

To prove that $\triangle ABC \cong \triangle A'B'C'$.

Proof. Place $\triangle A'B'C'$ in the position ABC'', thus making the side $A'B'$ coincide with its equal AB and causing the point C' to take up the position marked C''.

Join C and C'' by a straight line. Then, in $\triangle ACC''$, we

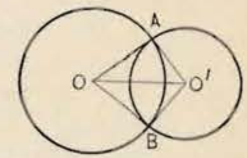
have	$AC = AC''$.	Given
Therefore	$\angle ACC'' = \angle AC''C$.	§ 40
Likewise, in $\triangle BCC''$, we have		
	$BC = BC''$;	Given
hence	$\angle BCC'' = \angle BC''C$.	Why?
Therefore	$\angle ACC'' + \angle BCC'' = \angle AC''C + \angle BC''C$,	Ax. 1 (§ 27)
that is,	$\angle ACB = \angle AC''B$;	
whence	$\triangle ACB \cong \triangle AC''B$,	§ 35
that is,	$\triangle ABC \cong \triangle A'B'C'$.	

46. Corollary 1. *Three sides determine a triangle; that is, if the three sides are given, the triangle is thereby fixed.*

The statement means that if the three sides of a triangle are known, any triangle made with these sides is congruent to any other triangle that has the same sides. See Problem 2, p. 3; Exs. 2, 3, p. 3; Ex. 11, p. 31.

EXERCISES

1. If three boards are nailed together in the form of a triangle, with one nail at each corner, will the frame thus made be quite stiff? Show how the corollary just stated is related to your answer.



2. Two circles whose centers are at O and O' intersect in A and B. Prove that $\triangle OAO' \cong \triangle OBO'$.

3. Prove that two triangles ABC and $A'B'C'$ are congruent if $AB = A'B'$, $BC = B'C'$, and the median through A equals the median through A'.

[HINT. Draw the two triangles. Let D be the point of intersection of the median through A with the side BC in $\triangle ABC$, and D' the corresponding point in $\triangle A'B'C'$. First show, by Theorem V, that $\triangle ADB \cong \triangle A'D'B'$, remembering that $BD = \frac{1}{2} BC$ and $B'D' = \frac{1}{2} B'D'$. Hence show that $\angle ADC = \angle A'D'C'$ by using 9, p. 29. Finally, show that $\triangle ADC \cong \triangle A'D'C'$. Then $\triangle ABC \cong \triangle A'B'C'$ by § 33.]

4. The framework of bridges, scaffolding, and other structures consists usually of a network of triangles whose sides are stiff pieces of iron or wood. Show, by means of Theorem V and Corollary 1, why the entire structure is stiff.

5. Show that the temporary bracing of a window frame in a building during its erection by a board nailed to the frame and to the floor, is an illustration of § 46.

6. Point out the triangle in the roof construction of Ex. 2, p. 43, which makes the roof structure rigid.

NOTE. These Exercises illustrate the great importance of Theorem V in structures of all kinds. The practical value of the study of triangles arises principally from the frequent application of Theorems I, II, and V, §§ 35, 37, 45, both in practical affairs and in the proofs of other geometric theorems. These three theorems are printed in boldface type to indicate their especial importance; they should be studied most carefully.

47. Theorem VI. *An exterior angle of a triangle is greater than either of the opposite interior angles.*

Given the $\triangle ABC$ having the exterior $\angle DBC$.

To prove that $\angle DBC > \angle A$; also that $\angle DBC > \angle C$.

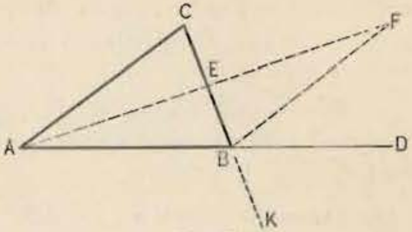


FIG. 35

Proof. Through E, the mid-point of BC, draw AE and prolong it to F, making $EF = AE$. Also, draw BF.

Then, in the $\triangle AEC$ and BFE we have

	$AE = EF$, and $BE = EC$,	Cons.
and	$\angle CEA = \angle BEF$.	10, § 31
Therefore	$\triangle AEC \cong \triangle BFE$;	Why?
whence	$\angle C = \angle FBE$.	Why?
But	$\angle DBE > \angle FBE$;	Ax. 10
hence	$\angle DBE > \angle C$,	Ax. 9

that is, $\angle DBC > \angle C$, which was to be proved.

[Similarly, by bisecting AB and proceeding as above it can be proved that $\angle ABK$, which equals $\angle DBC$ (why?), is greater than $\angle A$.]

EXERCISES

1. By drawing a series of triangles, determine what form a triangle tends to take as one of its exterior angles comes closer and closer in size to either of its opposite interior angles.

2. Prove that no triangle can have two right angles.

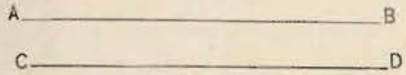
[HINT. If $\angle C$, Fig. 35, were equal to 90°, then $\angle DBC$ would be greater than 90° (§ 47); hence $\angle ABC < 90°$, since it is the supplement of $\angle DBC$. Write out the proof in full.]

3. Prove that no triangle can have two obtuse angles.

4. Prove that the bisector of an exterior angle of a triangle is perpendicular to the bisector of the adjacent interior angle.

PART II. PARALLEL LINES

48. Definitions. *Lines that lie in the same plane but do not meet however far they may be extended are said to be* ***parallel.***



Parallel Lines

FIG. 36

If two lines are cut by a third line, this third line is called a **transversal**, and the angles at the points of intersection are named as follows:

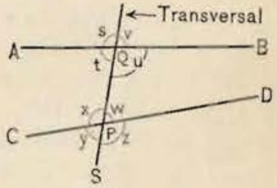


FIG. 37

t, u, x, and w are **interior** angles.

s, v, y, and z are **exterior** angles.

s and z, or v and y, are **alternate exterior** angles.

t and w, or u and x, are **alternate interior** angles.

v and w, or s and x, or u and z, or t and y, are **corresponding** angles.

49. Parallel Postulate. Besides the postulates of § 28, the following is necessary for the study of parallel lines:

Parallel Postulate. ***Only one line can be drawn through a given point parallel to a given line.*** That is, any other line so drawn will coincide with the first one.

This postulate was stated in a somewhat different form by **Euclid** (about 300 B.C.). It was recognized by him and by later mathematicians that it is peculiarly important. It is often called the Euclidean postulate.

50. Corollary 1. *Lines parallel to the same line are parallel to each other.* For, if two lines parallel to one and the same line should meet, there would be more than one parallel through a point, thus contradicting the preceding postulate.

51. Theorem VII. *When two lines are cut by a transversal, if the alternate interior angles are equal, the two lines are parallel.*

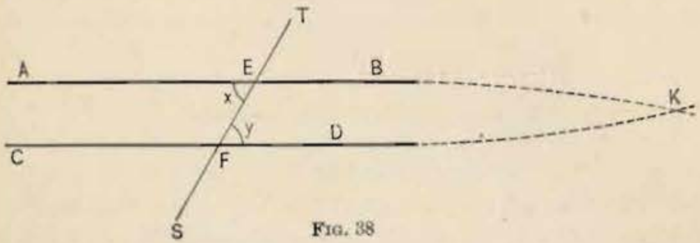


FIG. 38

Given any two lines AB and CD cut by the transversal ST at E and F, so that the alternate interior angles x and y are equal.

To prove that AB is parallel to CD.

Proof. Let us suppose for the moment that AB and CD are *not* parallel. In this case they would meet in some point which we will call K, and a triangle EFK would be formed. Then the exterior angle x of this triangle would be greater than its interior angle y, by Theorem VI. But this is impossible, since our hypothesis is that $\angle x = \angle y$. In other words, the conclusion that AB and CD are *not* parallel cannot be true if our hypothesis that $\angle x = \angle y$ is true. The only other possible conclusion is that AB and CD *are* parallel.

52. Corollary 1. *Lines perpendicular to the same line are parallel.*

EXERCISES

1. If several strips are nailed to a board at right angles to it, show that the strips are parallel.

2. If another board is nailed perpendicular to the strips of Ex. 1, show that it is parallel to the first board.

3. Draw any line CD (Fig. 38) and select any point E not on it. Draw any line through E to cut CD at some point, F. Lay off $\angle x = \angle y$ with a protractor, or as in § 7, p. 6. Show that *this process gives a line parallel to CD through E.*

53. The Indirect form of Proof. Argument by Reduction to an Absurdity. The student may have observed that the proof just given in § 51 differs essentially from other proofs we have given. In the proof of Theorem VII we meet for the first time what is called an **indirect proof,** otherwise known as a **reductio ad absurdum,** or **reduction to an absurdity.** This form of proof is of frequent occurrence. In substance, an indirect proof of a theorem consists in first supposing something *different* from the theorem's conclusion, and then showing that an absurdity results, thus leaving the theorem itself as the only possible statement of fact.

The use of the indirect proof is common, not only in geometry, but also in many of the familiar experiences of everyday life. Suppose, for example, that on a certain night a robbery is committed in a certain store and the next day Mr. A is suspected of having done it. He succeeds in proving, however, that throughout the night in question he was in another town. This is sufficient to free him of suspicion. Why? — Because the supposition that he *is* guilty is thus made to lead to the *absurd* conclusion that he was in two different places at the same time. The very strongest kind of argument is to show that some contention of an opponent leads to *absurd* conclusions.

EXERCISES

1. A closed wooden box is known to contain a piece of metal. A magnet is brought near and found to be attracted. The conclusion is drawn that the metal within the box is either iron or steel. Show that this conclusion is drawn by a process of indirect proof.

2. Using Theorem III, give an indirect proof of the following theorem: If no two of the three angles of a triangie are equal, the triangle cannot be isosceles.

3. Show that the reasoning used in § 50 is really a reduction to an absurdity.

54. Theorem VIII. ***If two parallel lines are cut by a transversal, the alternate interior angles are equal.***

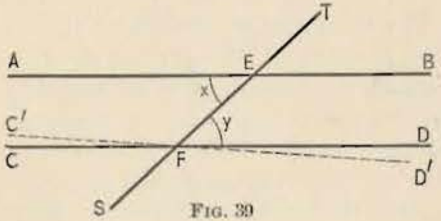


FIG. 39

Given the two parallel lines AB and CD cut by the transversal ST at E and F and making the alternate interior angles x and y.

To prove that the angles x and y are equal.

Proof. Suppose $\angle x \neq \angle y$ (For the symbol $\neq$, see p. 34.)

Now draw $C'D'$ through F, making $\angle D'FT = \angle x$.

Then	$C'D' \parallel AB$;	§ 51
but	$CD \parallel AB$.	Given

Thus we should have two different lines through the same point F parallel to the line AB, which is impossible, according to the postulate of § 49.

Therefore $\angle x = \angle y$, which was to be proved.

55. The Converse of a Theorem. The student may have observed that Theorems VII and VIII, though not the same, are very closely related. Careful examination of the two will show that the precise nature of this connection lies in the fact that the hypothesis and conclusion of the one have become, respectively, the conclusion and hypothesis of the other. In other words, the one is the other simply turned about. In general, when any two theorems are related in this way, the one is said to be the **converse** of the other.

Aside from geometry, there are many instances of statements and their converse. The following will illustrate this

fact, and also the fact that because a theorem or other statement is true we cannot always be certain that its converse is true.

(*Statement*) If a boat is made of wood, it floats. (*True*)

(*Converse*) If a boat floats, it is made of wood. (*False*)

An instance in Geometry in which the converse of a true statement is *false* is:

(*Statement*) If a figure is a square, it is a rectangle. (*True*)

(*Converse*) If a figure is a rectangle, it is a square. (*False*)

An instance in Geometry in which both the original and the converse theorems are *true* is:

(*Statement*) If two sides of a triangle are equal, the angles opposite them are equal. Theorem III, § 40.

(*Converse*) If two angles of a triangle are equal, the sides opposite them are equal. Theorem XVI, § 72.

In case both a theorem and its converse are true, they may be stated together in one sentence by properly using the clause "if and only if." Thus, Theorems VII and VIII may be combined into one as follows: "When two lines are cut by a transversal, the alternate interior angles are equal *if and only if* the lines are parallel."

The phrases "and conversely" or "and *vice versa*" are often used for the same purpose. Thus we might say, "If a body is lighter than water, it will float; *and conversely*." Or, "If a body is lighter than water, it will float; and *vice versa*."

EXERCISES

1. Make a true statement whose converse is false.

2. Make a true statement whose converse is true.

3. Write out the illustration you used in Ex. 2 in the form of a single sentence, using the clause "if and only if."

4. Rewrite each double statement on this page, using the phrase, (*a*) "if and only if"; (*b*) "and conversely," (*c*) "and *vice versa*." Which are true and which are false?

56. Theorem IX. *If two lines are cut by a transversal and the corresponding angles are equal, the lines are parallel.*

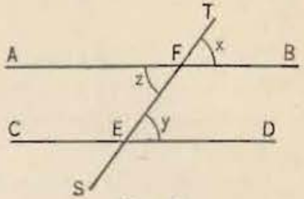


FIG. 40

Given AB and CD cut by the transversal ST in such a way that the corresponding angles x and y are equal.

To prove that AB is parallel to CD.

Proof.	$\angle x = \angle z,$	10, § 31
but	$\angle x = \angle y;$	Given
therefore	$\angle y = \angle z,$	Ax. 9
and hence also	$AB \parallel CD.$	§ 51

57. Corollary 1. *If two lines are cut by a transversal and the two interior angles on the same side of the transversal are supplementary, the lines are parallel.*

58. Corollary 2. *From a given point only one perpendicular can be drawn to a given line.* For, if there were two perpendiculars through the same point, they would be parallel (Why ?) but parallel lines cannot meet (§ 48). See also Ex. 2, p. 48.

EXERCISES

1. Two parallel pipes for hot and cold water lie flat along the same wall; at the end of each of them an elbow is screwed on which turns the pipe through a right angle. If the pipes connected to these elbows also lie flat against the same wall, will they be parallel? Connect your answer with §§ 56–58.

2. A rectangle (§ 24) has all its angles right angles; show that the opposite sides are parallel.

59. Theorem X. (*Converse of Theorem IX.*) *If two parallel lines are cut by a transversal, the corresponding angles are equal.*

Given the two parallel lines AB and CD cut by the transversal ST, making corresponding angles x and y.

To prove that $\angle x = \angle y$.

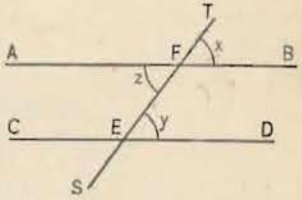


FIG. 41

Proof.	$\angle x = \angle z,$	Why ?
also	$\angle y = \angle z;$	Why ?
therefore	$\angle x = \angle y.$	Why ?

60. Corollary 1. *If a line is perpendicular to one of two parallels, it is perpendicular to the other also.*

EXERCISES

1. State and prove the converse of Corollary 1, Theorem IX.

2. The crosspieces (arms) which are put on a telephone pole to carry the wires are usually all perpendicular to the pole. How does this illustrate Theorem X or its corollary ?

3. Prove that the diagonals of all of the squares on a sheet of squared paper form continuous lines.

4. Prove that the bisectors of any pair of corresponding angles formed when a transversal cuts two parallel lines are themselves parallel. Is this true also for bisectors of alternate interior angles ?

5. Prove that the bisectors of any two interior angles formed when a transversal cuts two parallels are either parallel, or else perpendicular to each other.

6. Prove by §§ 56, 59 that *lines perpendicular to the same line* (*or to parallel lines*) *are parallel.* (See also § 52.)

61. Problem 1. *To construct a line parallel to a given line and passing through a given point.*

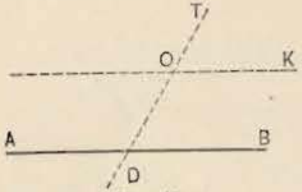


FIG. 42

Given the line AB and the point O.

Required to construct a line through O parallel to AB. (See Ex. 2, p. 50.)

Construction. Draw any line OT through O, cutting AB at some point, as D.

At O construct $\angle TOK = \angle ODB$. Prob. 5, p. 6.

Then the line OK (extended) will be the desired line through O parallel to AB.

Proof. Since $\angle TOK = \angle ODB$, the lines AB and OK are parallel. § 56

NOTE. The fact that in the problem of § 61, above, we have not only shown how to draw the desired line, but have afterwards proved that our method is correct, illustrates the principle that every construction problem should, in its solution, contain not only the construction, but also the proof of its correctness. This will be done hereafter in such problems as are worked out in the text, and the student should do the same in all construction problems that occur in exercises.

EXERCISES

1. Show how to construct a line parallel to a given line and passing through a given point, by means of § 52.

2. Show how to construct a line that makes one half a given angle with a given line at a given point.

[HINT. First bisect the given angle.]

3. Show how to construct a line that makes an angle of 45° with a given line at a given point.

4. (*a*) In order to draw a parallel to a line l through a point P, a draftsman will place a drawing triangle, or other object with two fixed edges, so that one side of the triangle

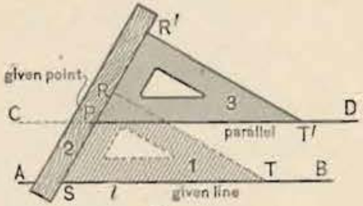


FIG. *a*

coincides with l, and the other side passes through P. He will then lay a ruler against the side of the triangle that passes through P, and finally slide the triangle along the edge of the ruler, until one corner of the triangle comes to P. Show that a line drawn along the side of the triangle, originally in coincidence with l, will be the parallel to l through P.

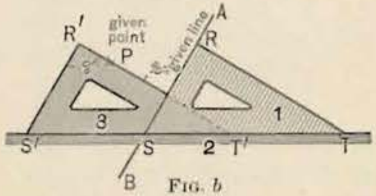


FIG. *b*

(*b*) Show how to draw, by means of a drawing triangle, a perpendicular to a given line AB through a given point. (Fig. *b*.) Notice that $\angle R = 90°$. See Ex. 3, p. 41.

5. Draw Fig. 38, omitting the portion dotted in the figure. Through some point C on CD construct a line parallel to ST. (The quadrilateral formed is called a *parallelogram*.)

6. A parallelogram (Ex. 5) is formed when one pair of parallel lines cuts another pair. Show that the sum of the two interior angles that have one side in common is 180°.

PART III. ANGLES AND TRIANGLES

62. Theorem XI. ***The sum of the three angles of any triangle is equal to two right angles, or 180°.***

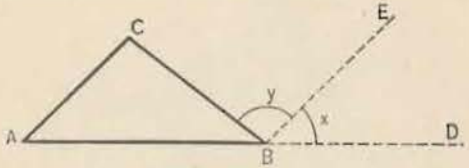


FIG. 43

Given any triangle ABC.

To prove that $\angle A + \angle B + \angle C = 2$ rt. $\angle s$.

Proof. Prolong AB to D and through B draw $BE \parallel AC$. Denote $\angle DBE$ by x, and $\angle EBC$ by y.

Then	$\angle x + \angle y + \angle ABC = 2$ rt. $\angle s$;	Why?
moreover	$\angle A = \angle x$,	§ 59
and	$\angle C = \angle y$.	§ 54
Therefore	$\angle A + \angle C + \angle ABC = 2$ rt. $\angle s$.	Ax. 9

NOTE. This famous theorem is of great practical importance. It was known by Pythagoras (about 500 B.C.). This figure was used by the Greek philosopher Aristotle (384–322 B.C.) and by the famous Greek geometrician Euclid (about 300 B.C.).

63. Corollary 1. ***The sum of the two acute angles of any right triangle is one right angle, or 90°.***

64. Corollary 2. ***An exterior angle of any triangle is equal to the sum of its opposite interior angles.*** That is, in Fig. 43

$$\angle CBD = \angle A + \angle C.$$

65. Corollary 3. *Each angle of an equilateral triangle is equal to 60°.*

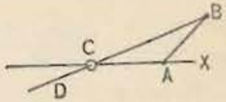
66. Corollary 4. *If two angles of one triangle are equal respectively to two angles of another triangle, then the third angles are likewise equal.*

EXERCISES

1. Two angles of a triangle are 10° 30′ and 85° 15′, respectively. What is the size of the third angle?

2. If the rafters of the roof represented in Ex. 2, p. 43, make an angle of 35° with the horizontal, show that the total angle at the ridge of the roof is 110°.

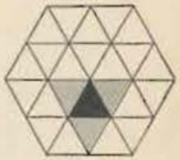
3. A crank AB is operated by means of a rod DCB, which slides through a ring at C. Show that the angle ACB is always half the angle XAB, provided $AC = AB$.



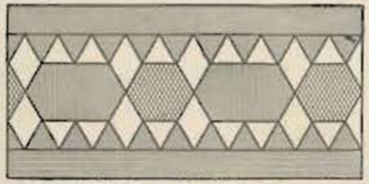
4. The exterior angles at A and C of a triangle ABC are 71° and 140°, respectively; how many degrees in the angle B?

5. Find the three angles of an isosceles triangle when one of the angles at the base is equal to one half the angle at the vertex.

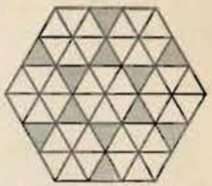
6. Draw an equilateral triangle, and draw three other equilateral triangles, placing one on each of the sides of the first one as a base. Show that if this process is repeated continually, the plane is divided into equilateral triangles that completely fill it. This fact is the basis for many interesting designs, some of which are shown below.



Orchard Plan



Tiled Floor Pattern



Linoleum Pattern

7. Show by § 62 that a triangle can have no more than one obtuse angle. Can it have more than one right angle? See Exs. 2, 3, p. 48.

67. Theorem XII. *Two angles whose sides are respectively parallel are either equal or supplementary.*

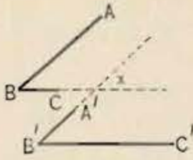


FIG. 44 *a*

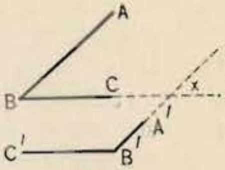


FIG. 44 *b*

There are two cases, as indicated by Figs. 44 *a* and 44 *b*.

CASE 1. **Given** ∠s B and B' with $AB \parallel A'B'$ and $BC \parallel B'C'$, as in Fig. 44 *a*.

To prove $\angle B = \angle B'$.

Proof. Prolong BC and $B'A'$ until they meet, thus forming the angle x.

Then $\angle B = \angle x$, Why?

and $\angle B' = \angle x$. Why?

Therefore $\angle B = \angle B'$. Why?

CASE 2. **Given** ∠s B and B' with $AB \parallel A'B'$ and $BC \parallel B'C'$, as in Fig. 44 *b*.

To prove that ∠s B and B' are supplementary.

[The details of the proof for this case are left to the student.]

EXERCISES

1. When in Theorem XII will the two angles be *equal?* when *supplementary?*

2. Show that Theorem XII is illustrated by the angles at the intersection of any two straight streets of uniform width.

3. Show that Theorem XII is illustrated by the angles at the intersection of two straight railroads.

4. A parallelogram (Ex. 5, p. 57) is a figure formed when one pair of parallel lines cuts another pair. Show by means of § 67 that the opposite interior angles are equal, and that the adjacent interior angles are supplementary.

68. Theorem XIII. *Two angles whose sides are respectively perpendicular to each other are either equal or supplementary.*

There are two cases, as indicated by Figs. 45 *a* and 45 *b*.

CASE 1. **Given** ∠s B and B' with $AB \perp B'C'$ and $BC \perp A'B'$, as in Fig. 45 *a*.

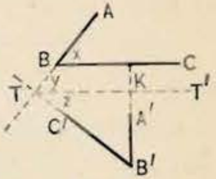


FIG. 45 *a*

To prove that $\angle B = \angle B'$.

Proof. Prolong AB and $B'C'$ until they meet at some point such as T, and through T draw a line $TT' \parallel BC$, meeting $A'B'$ prolonged at K.

Then $\angle x = \angle y$. Why?

Now $\angle z$ is the complement of $\angle y$. Def. § 18

Moreover, TKB' is a right angle; § 60

hence $\angle z$ is the complement of $\angle B'$. § 63

Therefore $\angle y = \angle B'$, 8, § 31

or $\angle B = \angle B'$. Ax. 9

CASE 2. **Given** ∠s B and B' with $AB \perp B'C'$ and $BC \perp A'B'$, as in Fig. 45 *b*.

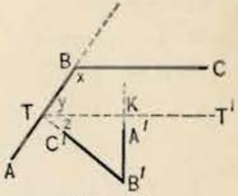


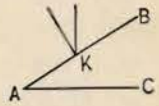
FIG. 45 *b*

To prove ∠s B and B' supplementary.

[The details of the proof for this case are left to the student.]

EXERCISES

1. When in Theorem XIII will the two angles be equal? when supplementary?



2. An object lies at a point K on an inclined plane AB. Show that the angle between the vertical line through K and a perpendicular to AB at K is equal to the angle CAB which the inclined plane makes with the horizontal.

69. Theorem XIV. ***Two right triangles are congruent if the hypotenuse and an acute angle of the one are equal respectively to the hypotenuse and an acute angle of the other.***

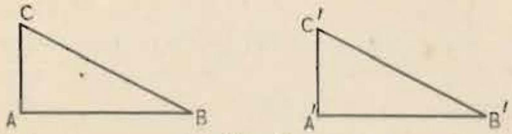


Fig. 46

Given the rt. $\triangle$ ABC and $A'B'C'$ with the hypotenuse $BC =$ hypotenuse $B'C'$ and $\angle B = \angle B'$.

To prove	$\triangle ABC \cong \triangle A'B'C'$.	
Proof. We have	$\angle A = \angle A'$ and $\angle B = \angle B'$.	Given
Hence	$\angle C = \angle C'$.	§ 66
Moreover	$BC = B'C'$,	Given
and therefore	$\triangle ABC \cong \triangle A'B'C'$.	Why ?

NOTE. Theorem XIV is frequently stated in the following form: *A right triangle is determined by its hypotenuse and one acute angle.*

EXERCISES

1. Draw an acute angle and then construct the right triangle containing this angle and having a hypotenuse 2 inches long.

2. Show that if two right triangles have one acute angle of one equal to one acute angle of the other, all of the angles of the one are equal to the corresponding angles of the other.

3. If one angle of a right triangle is 45°, show that the triangle is isosceles.

4. State and prove the converse of Ex. 3.

5. How could you construct (using only ruler and compasses) the right triangle one of whose acute angles is 60° and whose hypotenuse is a given length AB? (See § 65.)

70. Theorem XV. ***Two right triangles are congruent if the hypotenuse and a side of the one are equal respectively to the hypotenuse and a side of the other.***

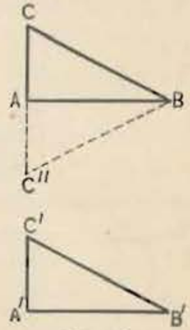


Fig. 47

Given the rt. $\triangle$ ABC and $A'B'C'$ with hyp. $BC =$ hyp. $B'C'$ and side $AB =$ side $A'B'$.

To prove that rt. $\triangle ABC \cong$ rt. $\triangle A'B'C'$.

Proof. Place $\triangle A'B'C'$ in the position ABC'' so that $A'B'$ coincides with its equal AB and C' falls at C'', opposite to C.

Then, the $\angle$ CAB and $C''AB$ being rt. $\angle$ (why ?), the line CAC'' will be a straight line. 12, § 31

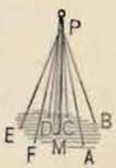
Now, in the $\triangle CC''B$, we have

	$BC = B'C''$;	Given
therefore $\triangle CC''B$ is isosceles;		
hence	$\angle ACB = \angle AC''B$,	§ 40
or	$\angle ACB = \angle A'C'B'$.	Ax. 9
Therefore	rt. $\triangle ABC \cong$ rt. $\triangle A'B'C'$.	§ 69

71. Corollary 1. *If two oblique lines of equal length are drawn from a point C in a perpendicular CD to a line AB* (Fig. 32, p. 42), *they cut off equal distances from the foot of the perpendicular, and conversely.*

EXERCISES

1. In the figure a mast is being held in a vertical position by means of a number of ropes (guy ropes) attached to the mast at the same distance from the ground. Show that the ropes will all have the same length if they are anchored at equal distances from the foot of the mast.



2. Construct a right triangle whose hypotenuse is 4 inches long and one of whose sides is 2 inches long.

72. Theorem XVI. [*Converse of Theorem III.*] *If two angles of a triangle are equal, the sides opposite are equal, and the triangle is isosceles.*

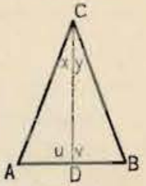


Fig. 48

Given the $\triangle ABC$ in which $\angle A = \angle B$.

To prove that $AC = BC$.

Proof. Draw CD bisecting $\angle C$.

Then, in the $\triangle$s ACD and BCD we have

	$\angle x = \angle y$ and $\angle A = \angle B$.	Why?
Therefore	$\angle u = \angle v$;	§ 66
but	$CD = CD$.	Iden.
Hence	$\triangle ACD \cong \triangle BCD$,	Why?
and therefore	$AC = BC$.	Why?

73. Corollary 1. *An equiangular triangle is also equilateral.*

74. Corollary 2. *If two oblique lines are drawn from a point C in a perpendicular CD to a line AB* (Fig. 48), *so as to make equal angles with AB, they are equal.*

75. Theorem XVII. *If two angles of a triangle are unequal, the sides opposite them are unequal and the greater side is opposite the greater angle.*

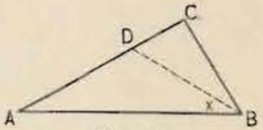


Fig. 49

Given the $\triangle ABC$ in which $\angle B > \angle A$.

To prove that $AC > BC$.

Proof. Draw BD meeting AC at D and making $\angle x = \angle A$.

Then, in the $\triangle ABD$

	$AD = BD$,	Why?
and	$BD + DC > CB$.	Post. 3
Therefore	$AD + DC > CB$;	Ax. 9
that is,	$AC > CB$.	

76. Corollary 1. *If two triangles have two sides of the one equal to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*

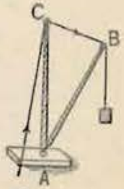
[Hint. In $\triangle$s ABC and $A'B'C'$, let $AB = A'B'$, $BC = B'C'$, and $\angle ABC > \angle A'B'C'$. Suppose $AB \leq BC$. Place $A'B'C'$ on ABC with $A'B'$ on AB. Join CC'. Then $\triangle BCC'$ is isosceles, and $\angle BC'C = \angle BCC'$. Hence, in $\triangle AC'C$, $\angle AC'C > \angle ACC'$, whence (§ 75) $AC > AC'$.]

Note. Corollary 1 is sometimes stated in the following brief form wherein it finds numerous illustrations in physics and mechanics. "The growth of an angle in a triangle means the growth of the side opposite it." It is to be understood, of course, in this statement, that as the angle is allowed to grow, the lengths of its including sides remain fixed.

77. Corollary 2. *If two oblique lines are drawn from a point C in a perpendicular CD to a line AB* (Fig. 48), *and if the base angles at A and B are unequal, the oblique line opposite the greater base angle is the greater;* in particular, *the perpendicular CD is itself the shortest line from C to any point of AB.*

EXERCISES

1. If, on account of some obstruction, one of the guy ropes mentioned in Ex. 1, p. 63, must be anchored nearer the foot of the mast than the others, show why that rope will be the shortest.



Crane

2. Is the string attached to a kite usually equal in length to the height of the kite above the ground? Connect your answer with § 75.

3. A simple form of crane consists of a beam AB hinged at A to a vertical mast AC and controlled by a wire rope attached at B and running over a pulley at C. When the rope is let out, the beam AB descends. Connect this fact with § 76.

78. Theorem XVIII. *If two sides of a triangle are unequal, the angles opposite them are unequal and the greater angle is opposite the greater side.*

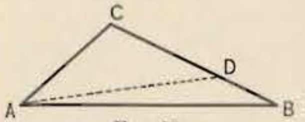


FIG. 50

Given $\triangle ABC$ in which $BC > AC$.

To prove that $\angle CAB > \angle B$.

Proof. On CB take $CD = AC$; draw AD.

Then, in the triangle ADC we have

	$\angle CAD = \angle CDA$.	Why?
But	$\angle CDA > \angle B$;	§ 47
therefore	$\angle CAD > \angle B$.	Why?
Moreover	$\angle CAB > \angle CAD$;	Ax. 10
therefore	$\angle CAB > \angle B$.	Why?

79. Corollary 1. *If two triangles have two sides of the one equal to two sides of the other, but the third side of the first greater than the third side of the second, then the included angle of the first is greater than the included angle of the second.*

[HINT. Prove by reduction to absurdity. Show first that equality of the included angles leads to a violation of § 35. Show that if the included angle of the second triangle is the greater, § 76 is violated.]

NOTE. Corollary 1 is sometimes stated in the following brief form wherein it finds numerous illustrations in physics and mechanics: "The growth of a side of a triangle means the growth of the angle opposite."

80. Corollary 2. *If from a point C in a perpendicular CD to a line AB* (Fig. 48) *unequal oblique lines are drawn to the base AB, the longer of the oblique lines is opposite the larger of the two base angles.*

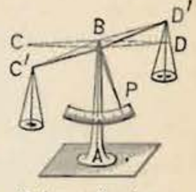
MISCELLANEOUS EXERCISES

1. Prove that the hypotenuse of a right triangle is its longest side.

2. By use of Theorem XVIII, prove that the perpendicular is the shortest line that can be drawn from a point to a straight line. (Compare § 77.)

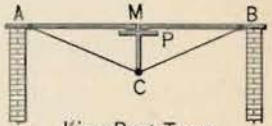
3. If the crank AB mentioned in Ex. 3, p. 59, is so arranged that $AC > AB$ (see the figure, p. 59), show that the angle ACB will always remain less than half the angle XAB during the rotation.

4. If the pans of a balance of the ordinary form shown in the figure are not precisely on the same level, show that each pan is nearer to the middle post than when the balance is level. Show also that the pans are always at equal distances from the middle post.



Balance Scales

5. A simple piece of bridge work consists of a frame like that shown in the adjoining figure, AM and MB being stiff pieces of steel merely hinged together at M, but the hinge resting on a plate P, which at some lower point C is connected to A and B by strong flexible wires. Show that a heavy weight may safely be put at M, even though the bridge is supported only by buttresses at A and B. (An arrangement of this kind is called an inverted King-Post truss.)



King-Post Truss

6. Determine the angles of a triangle when they are in the ratio 3 : 4 : 5.

7. If the exterior angle at A in a triangle ABC is 115°, and C is three times B, find B and C.

PART IV. QUADRILATERALS

81. Definitions. A plane figure bounded by four straight lines is called a **quadrilateral**. It is desirable to distinguish between several kinds of quadrilaterals as follows:

If two sides of a quadrilateral are parallel, it is called a **trapezoid**.

If each pair of opposite sides of a quadrilateral are parallel, it is called a **parallelogram**.

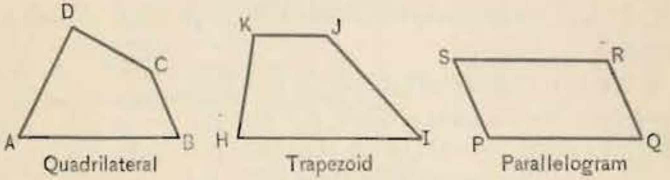


FIG. 51

A parallelogram all of whose angles are right angles is called a **rectangle**.

A quadrilateral all of whose sides are equal is called a **rhombus**.

A rectangle all of whose sides are equal is called a **square**.

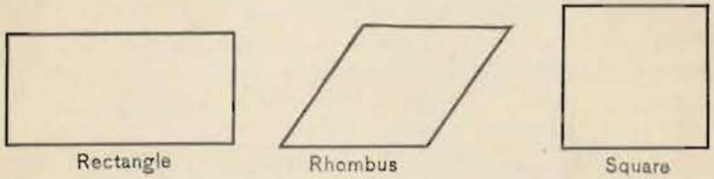


FIG. 52

The side upon which a quadrilateral appears to rest is called its **base**. Trapezoids and parallelograms, however, are considered as having *two* bases, one being the side upon which the figure appears to rest, and the other being the parallel side opposite it. Thus, in Fig. 53, AB and CD are *bases*.

The perpendicular distance between the bases (prolonged if necessary) of a trapezoid or parallelogram is called its **altitude**, as the line h in the figures below.

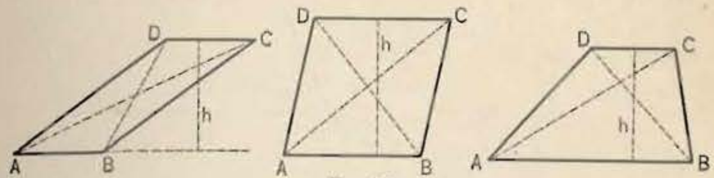


FIG. 53

A line joining opposite corners (vertices) of a quadrilateral is called a **diagonal**, as the line AC in Fig. 53.

In the figures above, the angles ABC and BCD are said to be *adjacent* to the side BC. A pair of angles such as A and C are said to be *opposite* angles of the quadrilateral.

EXERCISES

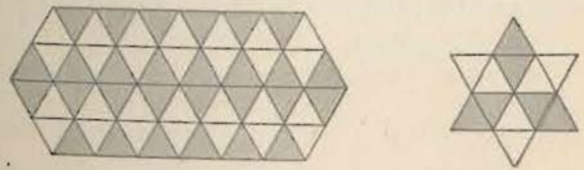
1. Prove that the angles adjacent to any side of a parallelogram are supplementary.

2. Prove that opposite angles of a parallelogram are equal.

3. If two adjacent angles of a parallelogram are in the ratio 17 : 1, how large is each angle of the parallelogram?

4. Construct the rhombus each of whose sides equals 2 inches and one of whose angles is 30°.

5. Show that two equilateral triangles that have a common



side, together form a rhombus. This is popularly called a "diamond," and is the basis of many designs.

82. Theorem XIX. ***Either diagonal of a parallelogram divides it into two congruent triangles.***

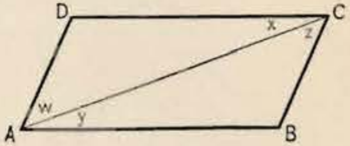


FIG. 54

Given the parallelogram $ABCD$ in which the diagonal AC has been drawn.

To prove that

$\triangle ABC \cong \triangle ADC.$

Proof. In the $\triangle$ ABC, ADC

we have	$AC = AC$,	
	$\angle x = \angle y$,	§ 54.
and	$\angle z = \angle w$;	Why?
therefore	$\triangle ABC \cong ADC$.	Why?

83. Corollary 1. *Any side of a parallelogram is equal to the side opposite it.*

84. Corollary 2. *The segments of parallel lines included between parallel lines are equal.*

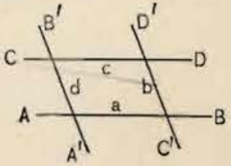


FIG. 55

[Thus, in Fig. 55, AB and CD are one pair of parallels, while $A'B'$ and $C'D'$ are another pair; they form the quadrilateral whose sides are represented by a, b, c, and d. Then, the corollary states that $a = c$ and $b = d$.]

EXERCISES

1. How does the ruled paper used in drawings in the Introduction (see § 25) provide an illustration of Corollary 2?

2. In the parallelograms that occur in the framework of bridges, a crosspiece is usually inserted along at least one of the diagonals. Why will this make the whole parallelogram stiff?

3. Cut a piece of paper in the form of a parallelogram and then cut it in two along one diagonal. Will the two triangles thus formed fit exactly upon each other? Why?

85. Theorem XX. *If a quadrilateral has both pairs of opposite sides equal, it is a parallelogram.*

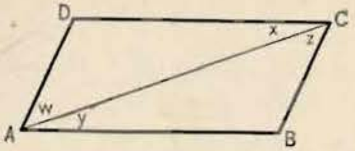


FIG. 56

Given the quadrilateral $ABCD$ in which $AB = DC$ and $BC = AD$.

To prove that $ABCD$ is a $\square$.

Proof. In the $\triangle$ ABC, ADC we have

	$AC = AC$, $AB = DC$, and $BC = AD$.	Why?
Therefore	$\triangle ABC \cong \triangle ADC$;	Why?
hence	$\angle x = \angle y$, and $\angle z = \angle w$.	Why?
It follows that	$AB \parallel DC$, and $BC \parallel AD$;	Why?
hence $ABCD$ is a $\square$.		§ 81

86. Theorem XXI. *If a quadrilateral has one pair of sides equal and parallel, it is a parallelogram.*

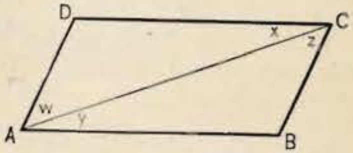


FIG. 57

Given the quadrilateral $ABCD$ in which AB is equal and parallel to DC.

To prove that $ABCD$ is a $\square$.

Proof. Draw the diagonal AC. Then in the $\triangle$ ABC and ADC, we have

	$AC = AC$, $AB = DC$, $\angle x = \angle y$;	Why?
hence	$\triangle ABC \cong \triangle ADC$.	Why?
Therefore	$\angle z = \angle w$, and hence $AD \parallel BC$.	Why?
But since $AB \parallel DC$, $ABCD$ is a $\square$.		Why?

87. Theorem XXII. *The diagonals of a parallelogram bisect each other.*

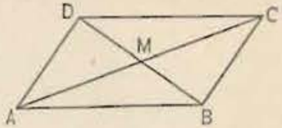


FIG. 58

Given the ▱ $ABCD$, and let its diagonals intersect at M.

To prove that $AM = MC$ and that $DM = MB$.

[HINT. Prove that $\triangle AMB \cong \triangle DMC$. Since the proof is easily carried out it is left to the student.]

88. Theorem XXIII. *Two parallelograms are congruent if two sides and the included angle of the one are equal respectively to two sides and the included angle of the other.*

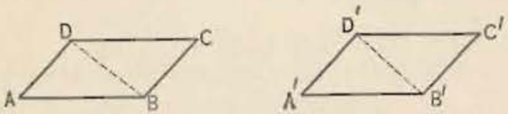


FIG. 59

Given the ▱ $ABCD$ and $A'B'C'D'$ in which $AB = A'B'$, $AD = A'D'$, and $\angle DAB = \angle D'A'B'$.

To prove ▱ $ABCD \cong$ ▱ $A'B'C'D'$.

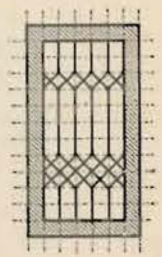
[HINT. Draw the diagonals DB, $D'B'$, and prove $\triangle ADB \cong \triangle A'D'B'$; also prove that $\triangle DCB \cong \triangle D'C'B'$. Then apply § 33.]

EXERCISES

1. Prove that the diagonals of a rectangle are equal.

2. Prove that the diagonals of a rhombus are perpendicular to each other.

3. Show that if each of the diagonals of all of the squares on a piece of squared paper are drawn, two new sets of continuous straight lines at right angles to each other are formed. This is the basis of many designs.



89. Theorem XXIV. ***The line joining the middle points of the two sides of a triangle is parallel to the base and equal to half the base.***

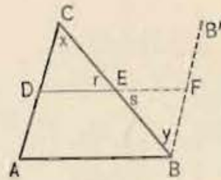


FIG. 60

Given the triangle ABC and the line DE joining the midpoints of the sides AC and BC.

To prove that $DE \parallel AB$, and that $DE = AB/2$.

Proof. Draw $BB' \parallel AC$ meeting DE prolonged at F.

Then, in the $\triangle$s DEC and EBF, we have

$$CE = EB, \angle r = \angle s, \angle x = \angle y. \qquad \text{Why?}$$

Therefore, $\triangle DEC \cong \triangle EBF$; hence, $DC = BF$. Why?

But $DC = AD$; hence, $BF = AD$, and $ABFD$ is a ▱. Why?

It follows that $DE \parallel AB$. § 81

The fact that $DE = AB/2$ *may now be established as follows:*

Since, as just shown, $ABFD$ is a ▱, we have

$$DF = AB. \qquad \text{Why?}$$

But

$$DF = DE + EF.$$

Moreover $DE = EF$, since $\triangle DEC \cong \triangle EBF$.

It follows that $DF = 2\,DE$, or $AB = 2\,DE$,

that is

$$DE = AB/2. \qquad \text{Ax. 4}$$

90. Corollary 1. (*Converse of § 89.*) *The line drawn through the middle point of one side of a triangle parallel to the base bisects the other side.*

[HINT. Draw the parallel; and draw the line connecting the middle points of the two sides. If these do not coincide, show by § 89 that § 49 is violated. For another proof, see Ex. 1, p. 75.]

91. Theorem XXV. *If three parallel lines cut off two equal portions of one transversal, they cut off two equal portions of any other transversal.*

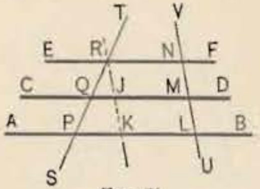


FIG. 61

Given the three parallel lines AB, CD, and EF; let ST be a transversal of which the two portions PQ and QR cut off by the three parallels, are equal.

To prove that the three parallels cut off equal portions LM and MN on any other transversal UV.

Proof. Through R draw a line $RK \parallel UV$ cutting CD at J.

Then	$KJ = LM$, and $JR = MN$;	§ 84
but	$KJ = JR$;	§ 90
hence	$LM = MN$.	Ax. 9

92. Corollary 1. *If a series of parallel lines cut off equal portions of one transversal, they cut off equal portions of any other transversal.*

FIG. 62

[HINT. Show that the portions cut off on any transversal are equal, taking two at a time, by Theorem XXV.]

93. Corollary 2. *If three parallel lines cut off two portions of one transversal, one of which is double the other, they cut off two portions of any other transversal, one of which is double the other.*

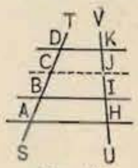


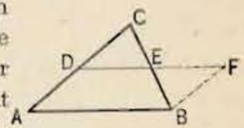
FIG. 63

[HINT. First draw a fourth parallel through the middle point of the larger portion of the first transversal, and then use Cor. 1, § 92.]

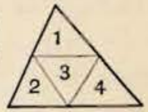
94. Corollary 3. *If three parallel lines cut off two portions of one transversal, one of which is n times the other, they cut off two portions of any other transversal, one of which is n times the other.*

EXERCISES

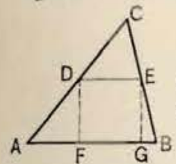
1. Prove that the line DE drawn through the middle point of one side of a triangle ABC parallel to the base bisects the other side by drawing $BF \parallel AC$ and showing that $\triangle DCE \cong \triangle FBE$. Compare § 90.



2. Prove that the lines joining the middle points of the three sides of a triangle divide it into four congruent triangles.

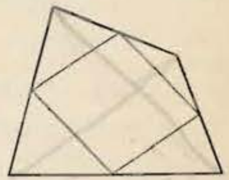


[HINT. Prove $\triangle 1 \cong \triangle 2 \cong \triangle 3 \cong \triangle 4$.]



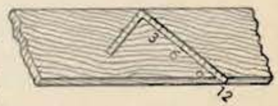
3. Prove that perpendiculars drawn from the middle points of two sides of a triangle to the third side are equal.

4. Prove that the lines joining the middle points of the sides of any quadrilateral form a parallelogram.



[HINT. Draw the diagonals of the original quadrilateral, and use § 89.]

5. A long board $7\frac{1}{2}$ in. wide is to be cut into 4 equal parallel strips. Show that it can be marked ready for sawing in the following manner:



Lay the corner (heel) of a carpenter's square on one edge of the board (see figure) and turn until the 12 mark is on the other edge. With an awl make dents at 3, 6, and 9. Move the square and repeat the operation. Then draw parallels through the dents thus made. Verify this process.

PART V. POLYGONS

95. Definitions. A plane figure bounded by any number of straight lines is called a **polygon**. The bounding lines are called **sides**; an angle between two adjacent sides, as the angle ABC in the figure, is called an **interior angle** of the polygon, while an angle between any one side and the adjacent side prolonged, as the angle CBK in the figure, is called an **exterior angle** of the polygon. In what follows, we assume that all polygons mentioned are **convex**, *i.e.* that each interior angle is less than 180°.

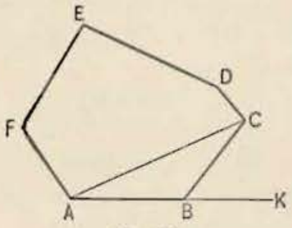


FIG. 64

A line joining any two non-consecutive vertices is a **diagonal**, as AC in the figure.

96. Kinds of Polygons.

A **triangle** is a polygon of three sides.

A **quadrilateral** is a polygon of four sides.

A **pentagon** is a polygon of five sides.

A **hexagon** is a polygon of six sides.

An **octagon** is a polygon of eight sides.

A **decagon** is a polygon of ten sides.

An **equilateral** polygon is one all of whose sides are equal.

An **equiangular** polygon is one all of whose interior angles are equal.

A **regular** polygon is one which is both equilateral and equiangular.

97. Theorem XXVI. *The sum of the interior angles of a polygon is two right angles taken as many times as the figure has sides, less two.*

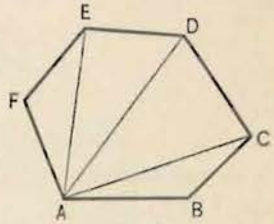


FIG. 65

Given the polygon $ABCD \cdots$ having n sides. [In Fig. 65, $n = 6$.]

To prove that the sum of its interior angles $= (n-2)$ 2 rt. ∠s.

Proof. Draw the diagonals, AC, AD, ⋯, dividing the polygon into $(n-2)$ △s.

The sum of the angles of the polygon is equal to the sum of the angles of these triangles. Ax. 11

But, the sum of the angles of any triangle = 2 rt. ∠s, § 62; hence the sum of the angles of $ABCD \cdots$ is $(n-2)$ 2 rt. ∠s.

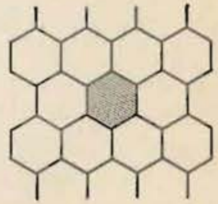
EXERCISES

1. What is the sum of the interior angles of a pentagon? a decagon? an octagon?

2. How many degrees in one angle of a regular pentagon? Answer the same question for a regular hexagon; regular decagon; regular octagon.

3. How many sides has the polygon each of whose exterior angles equals 30°?

4. Show that a regular hexagon can be made by placing six equilateral triangles with one vertex of each at the same point. (See Ex. 6, p. 59.)



5. Show that if a regular hexagon is drawn on each side of a given regular hexagon, the space in the plane around the given hexagon is just filled.

If the process is continued, show that the entire plane is divided into regular hexagons, in the manner of a honeycomb.

98. Theorem XXVII. *The sum of the exterior angles of a polygon formed by producing the sides in succession is equal to four right angles.*

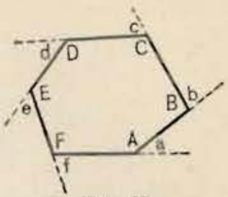


FIG. 66

Given the polygon $ABCD \cdots$.

To prove that the sum of its exterior angles $= 4$ rt. $\angle$s.

Proof. Denote the interior $\angle$s by A, B, C, D, $\cdots$ and the corresponding exterior angles by a, b, c, d, $\cdots$.

Then,
$$\angle A + \angle a = 2 \text{ rt. } \angle\text{s},$$
$$\angle B + \angle b = 2 \text{ rt. } \angle\text{s}.$$

In like manner the sum of each pair of angles at a vertex $=$ 2 rt. $\angle$s.

Therefore, the sum of both interior and exterior angles about the whole polygon will be 2 rt. $\angle$s taken as many times as the polygon has sides; that is, it will be $n \cdot 2$ rt. $\angle$s.

But the sum of the interior angles alone is $(n-2) \cdot 2$ rt. $\angle$s. § 97

Therefore, the sum of the exterior angles alone is
$$n \cdot 2 \text{ rt. } \angle\text{s} - (n-2) \cdot 2 \text{ rt. } \angle\text{s} = 2 \cdot 2 \text{ rt. } \angle\text{s} = 4 \text{ rt. } \angle\text{s}.$$

EXERCISES

1. What is the sum of the interior angles of a square? What is the sum of the exterior angles?

2. What is the sum of the interior angles of any quadrilateral? What is the sum of the exterior angles? Compare with Ex. 1.

3. What is the sum of the exterior angles of a pentagon? of a hexagon?

4. How large is each of the exterior angles of a regular pentagon? hexagon? decagon?

5. How many sides has the polygon the sum of whose interior angles equals the sum of its exterior angles?

PART VI. THE LOCUS OF A POINT

99. Locus of a point. In § 23 (Introduction) it is stated that a circle is a curve every point of which is equally distant from a point within (center). This definition may be stated in the following language: A circle is the *locus* (position) of all points equidistant from a given point. Using the same language, it may be said that the *locus* of all points equidistant from two parallel lines is the line lying midway between them, as the line EF in the figure.

FIG. 67

Similarly, the *locus* of all points common to two intersecting lines is simply one point; namely, their point of intersection.

NOTE. It is important to observe that in each of the preceding illustrations, *the locus not only* (1) *contains all points that satisfy a certain given condition, but it is also true that* (2) *there are no points on the locus that do* **not** *satisfy this condition.* Thus, in the figure above we can make the following two statements about EF: (1) EF contains all points equidistant from AB and CD; (2) there are no points on EF that are *not* equidistant from AB and CD.

Every true **locus** *possesses the properties* (1) *and* (2); hence in all locus problems two things are to be proved. This will be illustrated presently.

EXERCISES

1. What is the locus of all points 2 inches distant from a given straight line?

[HINT. Note that there are such points on *either* side of the given line.]

2. What is the locus of points 1 in. from a fixed point?

3. What is the locus of all points 4 inches distant from each of two given points which are 6 inches apart?

100. Theorem XXVIII. ***The locus of all points equidistant from the extremities of a line is the perpendicular bisector of that line.***

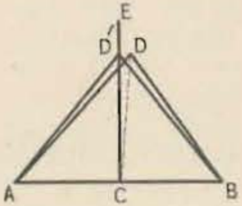


FIG. 68

Given the line whose extremities are A and B. Given also the line EC drawn $\perp$ AB at its middle point C.

To prove that EC is the locus of all points equidistant from A and B; that is (see § 99), to prove that

(1) any point D which is equidistant from A and B lies on EC,

(2) there is no point on EC not equidistant from A and B.

Proof. From D draw DA and DB.

Then	$\triangle ADC \cong \triangle BDC$.	§ 45
Therefore	$\angle ACD = \angle BCD$,	Why?
so that	$DC \perp AB$.	Why?

Hence D must lie on EC, which is the property (1) to be proved. 7, § 31

Again, let D' represent any point on the perpendicular bisector CE.

Then	$\triangle AD'C \cong \triangle BCD'$.	§ 35
Hence	$D'A = D'B$,	

which is the property (2) to be proved.

EXERCISE

1. What is the locus of the vertices of all isosceles triangles constructed on a given base?

101. Theorem XXIX. ***The bisector of an angle is the locus of all points equidistant from its sides.***

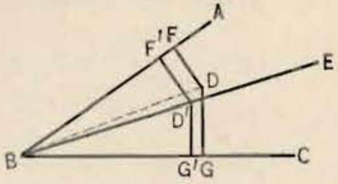


FIG. 69

Given the angle ABC and its bisector BE.

To prove that BE is the locus of all points equidistant from AB and BC; that is, to prove that

(1) any point D equidistant from AB and BC lies on BE,

(2) any point D' on BE is equidistant from AB and BC.

Proof. For the proof of (1) show that $\triangle BDF \cong \triangle BDG$.

For the proof of (2) show that $\triangle BD'F' \cong \triangle BD'G'$.

[The details of the proof are left as an exercise.]

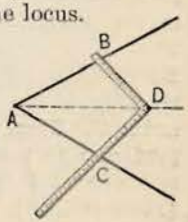
EXERCISES

1. By means of Theorem XXVIII prove the correctness of the construction given in § 5, p. 4. Similarly, prove the correctness of the constructions indicated in §§ 6, 8.

2. By means of Theorem XXIX prove the correctness of the construction given in § 9, p. 8.

3. What is the locus of a point that is equidistant from three given points? Show how to construct the locus.

4. A carpenter bisects an $\angle A$ as follows: Lay off $AB = AC$. Place a steel square so that $BD = CD$ as shown in the figure. Mark D and then draw the line AD. Show that AD bisects $\angle A$. Would this method be correct if the square were not a right angle at D?



102. Supplementary Theorems on Altitudes, Medians, etc.

Theorem XXX. *The perpendiculars erected at the middle points of the sides of a triangle meet in a point.*

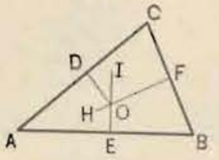


FIG. 70

Outline of proof. Let O be the point where the perpendicular bisectors EI, FH, of the sides AB, BC meet. Join O to the middle point D of AC. Prove that OD is then the perpendicular bisector of AC.

Theorem XXXI. *The bisectors of the angles of a triangle meet in a point.*

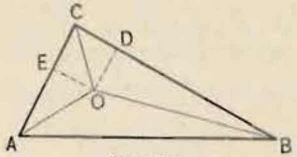


FIG. 71

Outline of proof. Draw the bisectors of $\angle A$ and B and suppose that they meet at O. Join O to the third vertex C and prove that rt. $\triangle OCD \cong$ rt. $\triangle OCE$, thus making $\angle ECO = \angle DCO$.

Theorem XXXII. *The altitudes of a triangle meet in a point.*

Outline of proof. Given $\triangle ABC$. Through its vertices draw lines parallel respectively to the opposite sides, forming $\triangle A'B'C'$. Then A is the mid-point of $B'C'$, since $BCAC'$ and $BCB'A$ are parallelograms. Similarly, C and B are mid-points of $A'B'$ and $A'C'$. Then, AD, BE, and CF are perpendicular to the sides of $A'B'C'$ at their mid-points and therefore meet in a point (Theorem XXX).

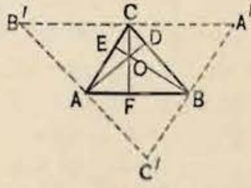


FIG. 72

Theorem XXXIII. *The medians of a triangle meet in a point which is two thirds of the distance from any vertex to the middle point of the opposite side.*

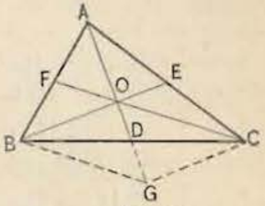


FIG. 73

Outline of proof. Draw the medians CF and BE and suppose they intersect at O. Draw AO and extend it to cut BC at D. Now draw BG parallel to FC, meeting AO (prolonged) at G. Join G and C. Then in the triangle ABG we have $AF = FB$. Therefore, $AO = OG$. (§ 90, p. 73.) Next prove $OE \parallel GC$. Then $BOCG$ is a parallelogram. Then $BD = DC$ (§ 87) which proves that the medians meet in a point.

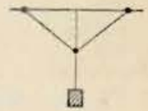
Moreover, $AO = OG$, while $OG = 2\ OD$ (§ 87). Whence, $AO = 2\ OD$. But $AD = AO + OD = 3\ OD$ so that $AO/AD = 2/3$; that is, $AO = \frac{2}{3} AD$.

MISCELLANEOUS EXERCISES ON CHAPTER I

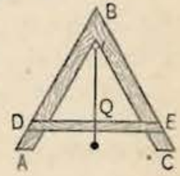
1. Given two angles of a triangle. Construct the third angle.

2. Prove that the bisectors of two supplementary adjacent angles are perpendicular to each other.

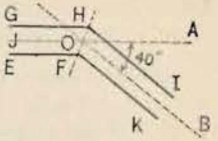
3. If a weight is hung from a small ring that slips freely on a cord, and if the cord is tied fast to two supports at equal heights, the ring will come to rest at the middle of the cord. Prove that the string by which the weight is attached then bisects the angle between the two portions of the cord.



4. A simple form of carpenter's level consists of three pieces of board nailed together in the shape of a capital letter A. A plumb bob is hung on a hook screwed at B. Show that any object upon which the feet A, C are set will be level in case the plumb line passes through the middle point Q of DE.



5. An angular joint for water pipes is to be constructed for two pipes that meet each other at an angle of 40°. If the seam FH is to make equal angles with the lines of centers; that is, if angle BOF = angle JOF, show that each of these angles must be taken equal to 70°.



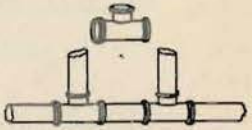
6. Prove that if two angles of a quadrilateral are supplementary, the other two are supplementary also.

7. What is the size of the obtuse angle formed between the bisectors of the acute angles of any right triangle?

8. Given a diagonal, construct the corresponding square.

9. Given the diagonals of a rhombus, construct the rhombus.

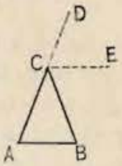
10. A so-called T joint for pipe is a piece made in the form of a letter T; the angle between the arms is accurately a right angle. Show, by § 52, that two pipes along the same wall joined to the same main pipe by T joints, are parallel.



11. Prove that the bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base.

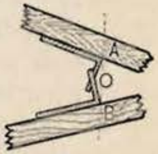
Given the $\triangle ABC$ with $AC = BC$ and $\angle DCE = \angle BCE$.

To prove $CE \parallel AB$.



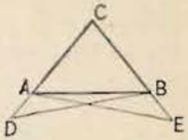
12. Prove (using Theorem I), that in an equilateral triangle the bisector of any angle forms two congruent triangles.

13. To cut two converging timbers by a line AB which shall make equal angles with them, a carpenter proceeds as follows: Place two squares against the timbers, as shown in the figure, so that $AO = BO$. Show that AB is then the required line.



14. Let ABC and RST be two congruent triangles. Prove that the medians drawn through A and R are equal. Prove also that the altitudes through A and R are equal.

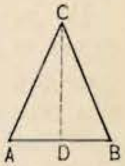
15. In the figure, $CA = CB$, $AD = BE$. Prove $\triangle ADB \cong \triangle ABE$.



16. In a right-angled triangle, if one of the acute angles is 30°, prove that the side opposite is half the hypotenuse.

17. Prove that the bisectors of the interior angles of a rectangle form a square.

18. If the bisector of an angle of a triangle is perpendicular to the side opposite the angle, the triangle is isosceles. Prove this statement.



19. Prove that the line joining the mid-points of the non-parallel sides of a trapezoid is equal to half the sum of the bases.

20. R is a river, and it is required to find the distance between the points B and A situated on the opposite shores. Show that this may be done (without crossing the stream) as follows:

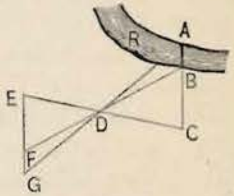
(1) Set a stake at some point C in line with AB.

(2) Set a second stake at some point D from which all three of the points A, B, C can be seen.

(3) Set stake at E in line with DC and such that $ED = DC$, and similarly a stake at F which shall be in line with DB and such that $FD = DB$.

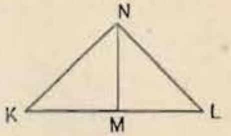
(4) Set stake at G in line with both EF and AD.

Then FG will be the required distance AB.

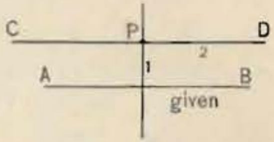


21. Prove that the angle between the bisectors of two angles of an equilateral triangle is double the third angle.

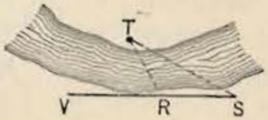
22. In the triangle KLN, NM is perpendicular to KL, and $KM = MN = ML$. Prove that KLN is an isosceles right triangle.



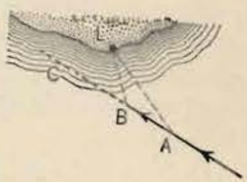
23. Show how to obtain (as a crease) a parallel to a given line through a given point on a piece of paper, by folding the paper, and prove that the result is correct.



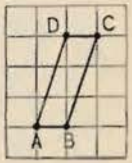
24. A man wishes to measure the distance between two points R and T on opposite sides of a stream. He takes a line VR and measures the angle TRV. He then walks along VR prolonged until he reaches a point S where $\angle TSR = \frac{1}{2} \angle TRV$. He then concludes $RS = RT$. Is he right? Why?



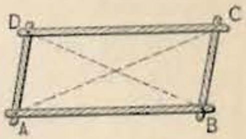
NOTE. The sailor uses this principle when he "doubles the angle on the bow" to find his distance from any object on shore. Thus if he is sailing in the direction ABC, and if L is a lighthouse, he measures the angle A, and if he notices when the angle that the lighthouse makes with his course is just twice the angle noted at A, then $BL = AB$. He knows AB from his log; hence he knows the distance BL.



25. Show that if the two ends of one side of one square on squared paper are connected by straight lines to the two ends of any parallel side of any other square of the same size, taken in the same order, a parallelogram is formed.

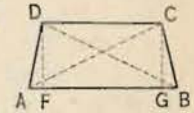


26. If a jointed frame of the form of a parallelogram is moved so that the angle at A grows less, show that (1) the angle at B increases; (2) the diagonal DB decreases; (3) the diagonal AC increases.



27. Can the frame of Ex. 26 be braced effectively by flexible wires along the diagonals? Why does this make the frame quite stiff? This principle is used in bridge building.

28. An isosceles trapezoid is one whose sides (other than the bases) are equal. Prove that the diagonals of an isosceles trapezoid are equal; also that its base angles are equal.



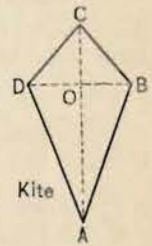
Given the trapezoid $ABCD$ in which $BC = AD$.

To prove (1) that $AC = BD$ and (2) that $\angle ABC = \angle BAD$.

[HINT. Draw the altitudes DF, CG, and prove (2) first.]

29. Show that if a rectangular door suspended on hinges hangs out of a vertical line, the bottom edge of door is out of a horizontal line. If the hinged edge leans away from the vertical line by 2 in. in every 3 ft. of its length, show that the bottom edge rises by 2 in. in every 3 ft. of its length.

30. A quadrilateral of which two pairs of adjacent sides are equal is called a **kite**. Show that one of the diagonals divides a kite into two congruent triangles; show that the other diagonal divides the kite into two triangles, each of which is isosceles.



31. Show that one pair of opposite angles of a kite are equal. If these angles are both right angles, show that the other two angles are supplementary.

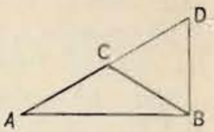
32. Construct a triangle having given the mid-points of its sides. See Theorem XXXII, § 102.

33. What is the locus of the middle points of all straight lines drawn from a fixed point to a fixed line of unlimited length?

34. Prove that if the diagonals of a parallelogram are equal and perpendicular to each other, the figure is a square.

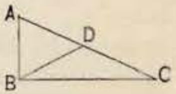
35. Construct an equilateral triangle, having given its altitude.

36. If in $\triangle ABC$, $AC = BC$, and if AC is extended to D so that $CD = AC$, prove that $DB \perp AB$. Hence show how to draw a perpendicular to a line AB at one end B without extending AB.



37. Show how to trisect (divide into three equal parts) a straight angle; a right angle; an angle of 45°.

38. ABC is a right-angled triangle. BD is drawn from the right-angle to the mid-point of the hypotenuse. Prove that the triangle ABC is thus divided into two isosceles triangles.



39. Prove that the sum of the sides of a quadrilateral is greater than the sum of its diagonals, but less than twice their sum.

40. Prove that the difference between the diagonals of a quadrilateral is less than the sum of either pair of opposite sides.

41. A line is terminated by two parallel lines. Through its mid-point any line is drawn terminated by the parallels. Prove that the second line is bisected by the first.

42. Prove that the perpendiculars drawn from the extremities of one side of a triangle to the median upon that side are equal.

CHAPTER II

THE CIRCLE

PART I. CHORDS. ARCS. CENTRAL ANGLES

103. Definitions. The **circle** (§§ 2, 23) is a curve, all points of which are equally distant from a point within, called the **center**; or (§ 99), it is the *locus* of all points equally distant from a given point.

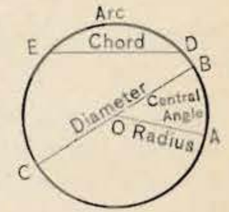


FIG. 74

By the definition of a circle, all its radii must be equal. (See §§ 2, 23.)

Any portion of the circumference is called an **arc** (§ 2).

One quarter of a circumference is called **a quadrant**.

A **chord** is a straight line joining the extremities of an arc.

A **diameter** is a chord that passes through the center.

The angle between any two radii is called a **central angle**.

In Fig. 74, the central angle AOB is said to **intercept** (*cut off*) the arc AB (written $\overset{\frown}{AB}$); while the arc AB is said to **subtend** the angle AOB.

An area bounded by two radii of a circle and the arc between them is called a **sector**.

An area bounded by a chord of a circle and its arc is called a **segment** of the circle.

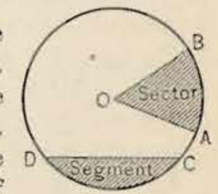


FIG. 75

Two circles are said to be *equal* when the radius of the one is equal to the radius of the other.

Two circles that have the same center are said to be **concentric**.

104. Postulates. In what follows we shall use the following facts as postulates:

(1) *As a central angle increases, its intercepted arc increases, and vice versa*; and *as a central angle decreases, its intercepted arc decreases, and vice versa.*

(2) *In the same circle (or equal circles), equal central angles intercept equal arcs*; and *equal arcs subtend equal central angles.*

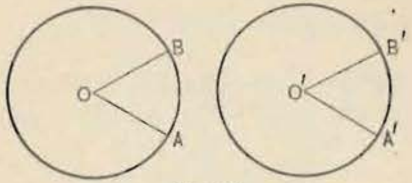


FIG. 76

Thus, in Fig. 76, the equal central angles AOB and $A'O'B'$ intercept the equal arcs AB and $A'B'$, and the equal arcs AB and $A'B'$ subtend the equal central angles AOB and $A'O'B'$.

105. Rotation. In considering the relations between the angles at the center of a circle and their intercepted arcs, it is helpful to think of the *rotation* of a wheel about its axle.

During such a rotation, any spoke of the wheel turns through a constantly increasing angle. The end of the spoke describes the arc of the circle that forms the rim of the wheel, while the angle described by the spoke intercepts on the rim precisely the arc described by the end of the spoke.

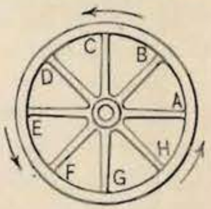


FIG. 77

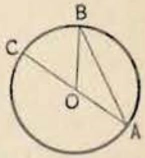
Thus, any two spokes of the same wheel describe equal angles in equal times. The arcs described by the ends of the two spokes are also equal. [(2), § 104.]

As the angle a spoke describes increases, the arc that its end describes also increases, that is, the greater of two angles at the center intercepts the greater arc. [(1), § 104.]

EXERCISES

1. What is the central angle between the hands of a clock when it is three o'clock? Answer the same question for four o'clock, eight o'clock, and half past nine.

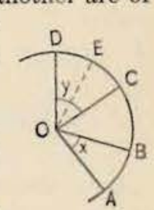
2. Show that the diameter of a circle is its greatest chord.



[HINT. In the figure, $AC = AO + OC = AO + OB$. But $AO + OB > AB$.]

3. Through a point within a circle to construct the longest possible chord.

4. Show that if one arc of a circle is double another arc of the same circle, the angle at the center subtended by the first is double that subtended by the second.

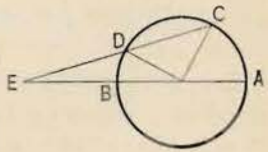


[HINT. First bisect the larger angle; then apply (2), § 104.]

5. Show that if one arc of a circle is n times another arc of the same circle, the angle at the center subtended by the first is n times that subtended by the second.

6. Show that if a circumference is divided into 360 equal arcs, the central angles subtended by these arcs are all equal (one degree); show that the number of degrees in any central angle is equal to the number of these small arcs contained in its intercepted arc.

7. Prove that two intersecting diameters divide a circumference into four arcs each of which is equal to one of the others.



8. The diameter AB and the chord CD are prolonged until they meet at E. Prove that $EA > EC$ and $EB < ED$.

106. Theorem I. *In the same circle or in equal circles, equal arcs subtend equal chords.*

Given the two equal ⊙ O and O' in which $\widehat{AB} = \widehat{A'B'}$.

[⊙ is the symbol for *circle*; ⌒ is the symbol for *arc*.]

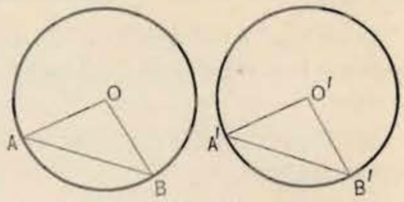


FIG. 78

To prove that chord AB = chord $A'B'$.

Proof. Draw the radii OA, OB, $O'A'$, $O'B'$. Then in the △ AOB, $A'O'B'$, we have

$OA = O'A'$, $OB = O'B'$, § 103

and $\angle O = \angle O'$. § 104

Therefore $\triangle AOB \cong \triangle A'O'B'$; § 35

hence $AB = A'B'$.

107. Theorem II. (*Converse of § 106.*) *In the same circle or in equal circles, equal chords subtend equal arcs.*

Given the two equal ⊙ O and O' (Fig. 78) in which the chord AB = the chord $A'B'$.

To prove that $\widehat{AB} = \widehat{A'B'}$.

Outline of proof. Draw the radii OA, OB, $O'A'$, $O'B'$. Then show, by § 45, that $\triangle AOB \cong \triangle A'O'B'$ and apply § 104.

EXERCISES

1. Show that in the same or equal circles the greater of two arcs subtends the greater chord, and *vice versa*. [Use § 104 (1) and § 76.]

2. Show that the construction of an angle equal to a given angle (§ 7) illustrates § 107. Hence prove that the construction in § 7 is correct.

3. State Theorems I and II, using the phrase (*a*) "and conversely"; (*b*) "and *vice versa*"; (*c*) "if and only if." See § 55.

108. Theorem III. *A diameter perpendicular to a chord bisects the chord and the arc subtended by it.*

Given the diameter $DF \perp$ chord AB at K.

To prove that $AK = KB$ and that $\widehat{AF} = \widehat{FB}$.

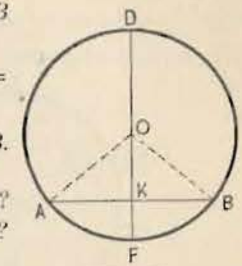


FIG. 79

Proof. Draw the radii OA and OB. Then

$OK = OK$, and $OA = OB$; Why?

hence rt. $\triangle OKA \cong$ rt. $\triangle OKB$. Why?

Therefore

$AK = KB$, and $\angle AOK = \angle BOK$; Why?

hence $\widehat{AF} = \widehat{FB}$. (2) § 104

NOTE 1. In the figure above the chord AB may be regarded as subtending not only the arc AFB, but also the larger arc ADB. It is customary to speak of the two as the **minor** arc and the **major** arc corresponding to the given chord. Similarly, any central angle subtends both a minor and a major arc. Unless otherwise stated, the minor arc is the one to be understood hereafter in any statement where both might play a part.

NOTE 2. This *extremely important*, though simple, figure (Fig. 79), occurs in the greatest variety of practical affairs, and in many geometric theorems and constructions. (See §§ 5, 6, 8, 9, 40, 43, 44, 72, 100, 102, and Exs. 2, p. 43; 1, p. 45, etc.)

EXERCISES

1. Prove that a diameter perpendicular to a chord bisects the major arc subtended by it.

2. What is the locus of the mid-points of a system of parallel chords?

3. Prove that the perpendicular bisector of a chord passes through the center of the circle and bisects the arcs (major and minor) subtended by the chord.

109. Theorem IV. *In the same circle or in equal circles, equal chords are equally distant from the center; and, conversely, chords that are equally distant from the center are equal.*

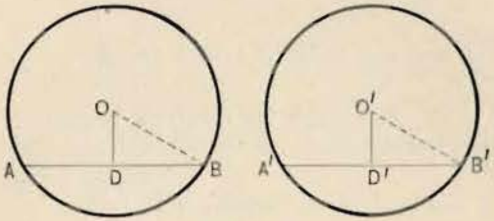


FIG. 80

Given the equal chords AB and $A'B'$ in the equal ⊙ O and O'.

To prove that AB and $A'B'$ are equally distant from the centers O and O', respectively.

Proof. Draw $OD \perp AB$ and $O'D' \perp A'B'$; draw also the radii OB and $O'B'$.

Prove that $\triangle DOB \cong \triangle D'O'B'$,

and hence $OD = O'D'$.

In the *converse*, it is given that the chords AB and $A'B'$ are equally distant from the centers of the equal ⊙ O and O'.

To prove that $AB = A'B'$.

Proof. Show that $\triangle DOB \cong \triangle D'O'B'$

and hence that $DB = D'B'$.

But $AB = 2\,DB$ and $A'B' = 2\,D'B'$; § 108

whence $AB = A'B'$. Why?

EXERCISES

1. Show that two boards sawed from the same log, or from equal logs, at equal distances from the center, are equal in width.

2. What is the locus of the mid-points of a system of equal chords in a circle?

110. Theorem V. *In the same circle, or in equal circles, if two unequal chords are drawn, the longer one is nearer the center.*

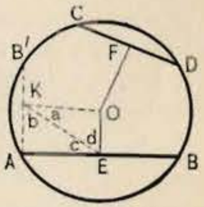


FIG. 81

Given the ⊙ O with the two chords AB and CD such that $AB > CD$. Also, let OE and OF be the perpendicular distances from O to AB and CD, respectively.

To prove that $OE < OF$.

Proof. From A lay off the chord $AB' = CD$. Then draw the perpendicular OK, and finally join K to E by the line KE.

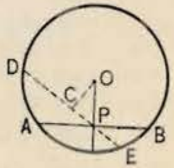
Since	$AE = \frac{1}{2}AB$ and $AK = \frac{1}{2}AB'$,	§ 108
it follows that	$AE > AK$, for $AB > AB'$.	Given
Hence	$\angle b > \angle c$,	§ 78
and consequently	$\angle a < \angle d$.	Ax. 8
Therefore	$OE < OK$.	§ 75
But	$OF = OK$.	§ 109
Therefore	$OE < OF$.	Ax. 9

111. Corollary 1. (*Converse of § 110.*) *In the same circle or in equal circles, if two chords are unequally distant from the center, the more remote is the less.*

EXERCISES

1. Show that a board sawed from a circular log is wider than another board sawed from the same log at a greater distance from the center.

2. Show that the least chord that can be drawn through a given point within a circle is the chord perpendicular to the radius through that point.



[HINT. Draw any other chord through P and draw OC perpendicular to it. First prove $OP > OC$.]

3. Show that a diameter is longer than any other chord.

112. Chords. The relations between chords, arcs, and central angles of the same circle appear vividly in connection with rotation (§ 105). Thus the chord that subtends the arc grows as the angle grows; that is, as the wheel rotates, until the chord reaches its greatest possible size, the diameter of the circle. At this time, the angle the spoke has described is 180°, or a straight angle. As the rotation goes on, the chord shrinks again to less than the length of the diameter, until the wheel has made one *complete revolution*, when the chord has shrunk to zero.

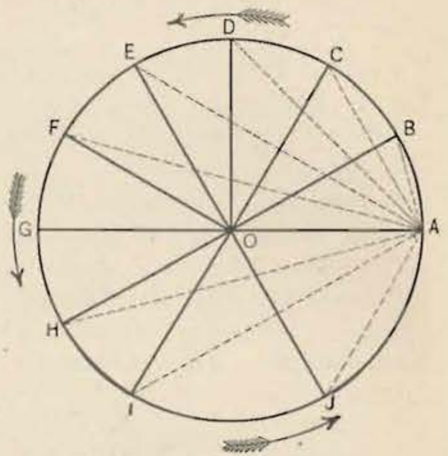


FIG. 82

If a circle of known radius is drawn, any central angle subtends a chord of some definite length. Taking the radius as 1 unit, the lengths of the chords corresponding to various central angles for every degree from 0° to 90° are given in the **table of chords**, TABLES, pp. iii–vii.

By means of this table, any angle can be laid off from any point as vertex, by drawing a circle of unit radius about it.

Thus, the chord that corresponds to a central angle of 29° is very nearly equal to $\frac{1}{2}$. To lay off 29° at a point P on a line MN, draw a circle x of unit radius about P as center, cutting MN at Q, as in § 7, p. 6. Then draw an arc y about Q as center with radius $\frac{1}{2}$. If x and y intersect at O, $\angle OPQ = 29°$, approximately.

Let the student lay off the angles of 31°, 69°, etc., in a similar manner, using the Tables.

113. Angular Speed. The facts about the rotation of wheels or other parts of machinery are often clearly expressed in terms of the speed with which the wheel (or other part) is rotating. This speed of rotation means the amount of angle through which the wheel turns in one unit of time; for example, we say that a certain wheel is turning at the rate of four revolutions per minute, or that some other wheel is rotating 50° per second. Such a speed of rotation of a wheel is called its *angular speed*.

Which of the two wheels just mentioned is moving the faster? The answer to this question is found by changing revolutions per minute into degrees per second as follows: The first wheel is rotating four revolutions per minute. Since four revolutions means 1440°, that wheel is making 1440° per minute, or 1440° every 60 seconds. Dividing by 60, we find that it is going 24° each second. Hence the second wheel is rotating over twice as fast.

EXERCISES

1. Prove that in the same circle, or in equal circles, equal chords subtend equal central angles, and *vice versa*.

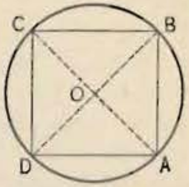
2. Prove that the greater of two chords of a circle subtends the greater central angle, provided the central angle less than 180° is understood in each instance.

3. Draw a circle of radius one inch. Measure approximately the length of the chord of each of the following angles: 45°, 60°, 75°, 90°, 120°, 180°, 210°. Check, when possible, by means of the tables, pp. iii–vii.

4. If an angle is doubled, is its chord doubled? Compare the lengths of chords of angles of 45° and 90°. Compare the lengths of chords of angles of 60° and 180°. Check your results by means of the tables on pp. iii–vii.

5. If one wheel is rotating through 5 revolutions per minute, and another through 30° per second, which is rotating the more rapidly?

MISCELLANEOUS EXERCISES. PART I



1. Show that a chord equal to the radius subtends an angle of 60°.

2. If a circle is drawn through all four vertices of a square, show that the arcs intercepted by its sides are equal. Prove that each arc subtends a central angle of 90°.

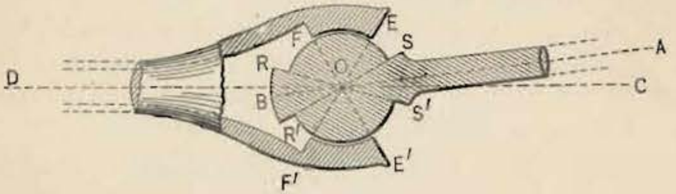
3. Show that if the vertices of an equilateral triangle all lie on a circle, each side intercepts an arc of 120°.

4. Show that if the vertices of an equilateral hexagon all lie on a circle, each side intercepts an arc of 60°. Show how to construct such a figure.

5. If a wheel is rotating two revolutions per minute, what part of a revolution does it make in one second? How many degrees does it turn through in one second?

6. The earth revolves once in 24 hours. How many degrees correspond to one hour? *Ans.* 15°.

7. A jointed extension rod — such as that used on desk lights — is made by having a piece in the form of a circular ring fit over the end of another circular piece, in the manner illustrated



in the figure. An excessive motion is prevented by projections at the points R and R' and at S and S'.

Show that when the stop S has reached E, the line AB has turned away from the line DC by half the difference between the angles ROS and EOF.

PART II. TANGENTS AND SECANTS

CIRCUMSCRIBED AND INSCRIBED TRIANGLES

114. Definitions. A line of indefinite length which cuts a circle is called a **secant**.

A line of indefinite length which touches a circle in but **one** point is called a **tangent**; this point P is then called the **point of contact**, or **point of tangency**.

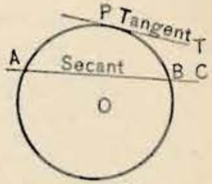
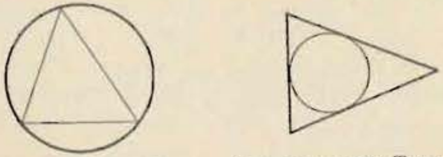


FIG. 83

A triangle or other polygon is said to be **inscribed** in a circle when its vertices all lie on the circumference. Under the same conditions, the circle is said to **circumscribe** the polygon.



INSCRIBED TRIANGLE CIRCUMSCRIBED TRIANGLE

FIG. 84

A triangle or other polygon is said to be **circumscribed** about a circle when its sides are all tangent to the circle. Under the same conditions, the circle is said to be **inscribed** in the polygon.

A good idea of a line tangent to a circle is obtained by placing a coin (representing a circle) against the edge of a ruler (representing a straight line).

A tangent to a circle may be roughly drawn by placing a ruler so that its edge just meets, but does not cut across, the circumference. If several tangents are drawn to the same circle in this manner, and extended to meet, a circumscribed polygon results.

115. Theorem VI. ***A line perpendicular to a radius at its extremity is tangent to the circle.***

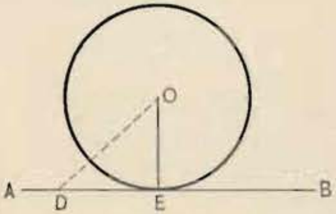


FIG. 85

Given the $\odot O$ and the line $AB \perp$ to the radius OE at its extremity.

To prove that AB is a tangent to the circle.

Proof. Take any point D on AB except E and draw OD.

Then $OD > OE$. § 77

Therefore the point D is not on the circle; § 103
that is, no point of AB except E lies on the circle.

Whence AB is a tangent. § 114

116. Corollary 1. *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*

[Note that we know that $OE < OD$. Then apply § 77.]

117. Corollary 2. *A perpendicular to a tangent at its point of contact passes through the center of the circle.*

[HINT. Draw the radius to the point of contact and apply Corollary 1, together with § 58.]

118. Theorem VII. *Two tangents drawn to a circle from a point outside are of equal length.*

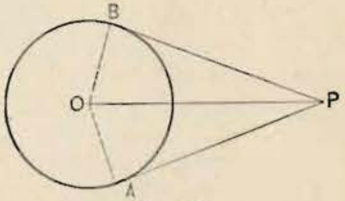


FIG. 86

[The proof is left to the student, with the aid of Fig. 86.]

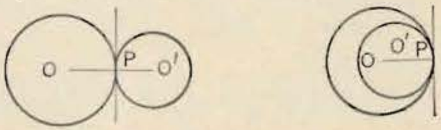
EXERCISES

1. Prove that two tangents drawn to a circle at the extremities of a diameter are parallel.

2. Construct a tangent to a circle parallel to a given line.

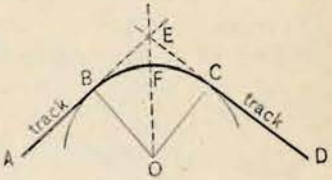
3. Draw two concentric circles (§ 103) and prove that all chords of the greater circle that are tangent to the smaller circle are equal.

4. Let O and O' be two circles which are *tangent* to each other; that is, which have but one point in common. Prove (1) that the line joining O and O' (line of centers) passes through the point common to the two circles (see 12, § 31); and (2) that a perpendicular to the line OO' *at* the common point



P will be a tangent to both circles. Consider the case in which the circles are tangent *externally*, and also the case in which they are tangent *internally*, as indicated in the figures.

5. A railroad curve joining two pieces of straight track AB and CD, is usually a circular arc tangent at B and C to AB and CD, respectively. Show that the center O of the circular arc is the intersection of the perpendiculars to AB and CD at B and C, respectively.



6. If AB and CD (Ex. 5) are extended to meet at E, show that EO bisects the angle at O. Hence show that F, the intersection of EO and $\widehat{BC}$, is the center of $\widehat{BC}$.

7. Show that the two radii and the tangents at their extremities (Fig. 86, or figure for Ex. 5) form a *kite*; that is (Ex. 30, p. 87), two pairs of its adjacent sides are equal.

119. Theorem VIII. *Two parallel lines intercept equal arcs on a circle.*

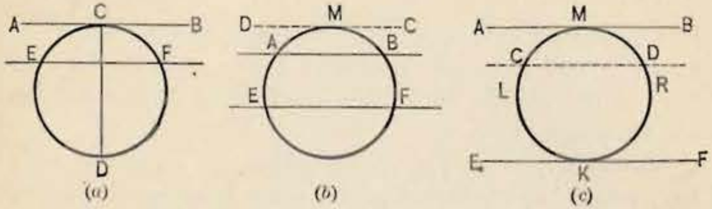


FIG. 87

CASE I. *When the parallels are a tangent and a secant.*

Given the tangent $AB \parallel$ the secant EF (Fig. 87 a); also, let C be the point of contact of AB.

To prove that $\widehat{EC} = \widehat{CF}$.

Proof. Draw the diameter CD. Then $CD \perp AB$; § 116

hence $CD \perp EF$, Why?

and therefore $\widehat{EC} = \widehat{CF}$. § 108

CASE II. *When the parallels are both secants.*

Given the parallel secants AB and EF (Fig. 87 b).

To prove $\widehat{BF} = \widehat{AE}$.

Proof. Draw $DC \parallel AB$ and tangent to the circle. Let M be the point of contact.

Then $DC \parallel EF$. Why?

Whence $\widehat{FM} = \widehat{EM}$, and $\widehat{BM} = \widehat{AM}$. Case I

Therefore $\widehat{BF} = \widehat{AE}$. Ax. 2

CASE III. *When the parallels are both tangents.*

Given AB and EF parallel tangents touching the circle at M and K, respectively (Fig. 87 c).

To prove that $\widehat{MLK} = \widehat{MRK}$.

Outline of proof. Draw $CD \parallel AB$ (Fig. 87 c); and prove $\widehat{MC} = \widehat{MD}$, and $\widehat{KC} = \widehat{KD}$. Then apply Ax. 1.

EXERCISES

1. Show that a tangent parallel to any chord of a circle has its point of tangency in the center of the arc that the chord intercepts.

2. State and prove the converse of Theorem VIII, § 119.

[HINT. (Case I): Given $\widehat{CF} = \widehat{CE}$; to prove $AB \parallel EF$.

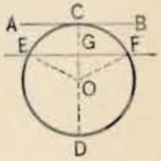
Draw the diameter CD, also the radii OE, OF.

Then $CD \perp AB$. (Why?)

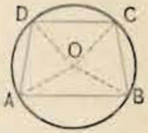
Now, since $\widehat{CF} = \widehat{CE}$, we have $\angle COF = \angle COE$. (Why?)

Therefore $\triangle FGO \cong \triangle EGO$ (Why?), so that $\angle FGO = \angle EGO$.

Therefore $CD \perp EF$. Since $CD \perp AB$ and $CD \perp EF$, we have $AB \parallel EF$ as desired.]

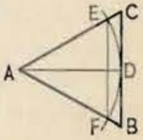


3. If every vertex of a trapezoid lies on a circle, prove that it is isosceles; that is, that $BC = AD$.



4. If a thick board is sawed out of a circular log, the circular arc that forms one edge is equal to that which forms the other edge. Connect this fact with § 119.

5. Let ABC be an isosceles triangle, with $\angle B = \angle C$; and let AD be its altitude from A to BC. Draw a circle with center at A and with radius AD, cutting the sides AB and AC in F and E, respectively. Show that $EF \parallel BC$.



6. Draw a secant intersecting two concentric circles and prove that the portions intercepted between the two circles are equal.

7. Prove that if every vertex of a parallelogram lies on a circle, any two opposite sides are equidistant from the center.

8. Prove that if a polygon is inscribed in a circle, the perpendicular bisectors of the sides meet in a point.

120. Theorem IX. *Through three given points not all on the same straight line, one and only one circle can be drawn.*

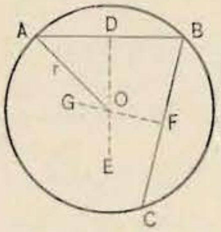


FIG. 88

Given the three points A, B, and C not all on the same straight line.

To prove that one and only one circle can be passed through A, B, and C.

Proof. The locus of all points equally distant from A and B is the perpendicular bisector DE of the line AB, while the locus of all points equally distant from B and C is the perpendicular bisector FG of the line BC. § 100

Therefore the intersection O of DE and FG is equally distant from all three of the points A, B, and C.

Hence, the circle drawn with O as center and with a radius equal to the line AO will pass through A, B, and C.

That this is the *only* such circle follows from the fact that the lines DE and FG can intersect in but one point. 4, § 31

NOTE. Theorem IX is frequently stated in the following brief form: *Three points determine (fix) a circle.* The proof also shows how to **construct the circle passing through three given points.** Thus, given A, B, and C (Fig. 88); draw AB and BC and erect their perpendicular bisectors, DE and FG. The intersection O of DE and FG is the center of the desired circle; its radius is one of the equal distances OA, OB, or OC.

121. Corollary 1. *A circle may be drawn to circumscribe any triangle.*

[Draw the three perpendicular bisectors of the sides of the triangle.]

122. Corollary 2. *The perpendicular bisectors of the sides of a triangle meet in a point.* (Compare Theorem XXX, § 102.)

This point is called the **circumcenter**, because it is the center of the circumscribed circle.

123. Corollary 3. *A circle may be completed if any arc of it is given.*

[HINT. Take three points on that arc and draw a circle through them.]

EXERCISES

1. Try to prove Theorem IX, § 120, for three points A, B, C, that lie on a straight line. At what place does the proof break down, and why?

2. Draw a triangle of any shape and then construct the circle which passes through its vertices.

3. How many circles can be drawn through *two* given points? What is the locus of the centers of all such circles?

4. How many circles *having a given radius* can be drawn through two given points? Is it always possible to have *one* such circle? Why?

5. Show how to construct the center of a given circle.

[HINT. Erect the perpendicular bisectors of any two chords.]

6. Show that a circle can be circumscribed about a given quadrilateral if (and only if) the perpendicular bisectors of the sides all meet in a single point.

7. Show that a circle may be circumscribed about any square; any rectangle.

8. Show that the perpendicular bisectors of the opposite sides of a parallelogram are parallel unless they coincide. Hence show that an inscribed parallelogram is a rectangle.

124. Theorem X. *A circle may be* **inscribed** *in any triangle.*

Given the triangle ABC.

To prove that a circle may be inscribed in it.

Proof. Draw the bisectors of the angles A, B, C; *these three bisectors meet in a point I.* § 102

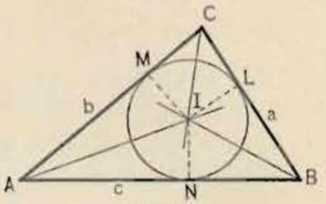


Fig. 89

Draw the perpendiculars IL, IM, IN, to the three sides a, b, c, respectively; then

$$IL = IM = IN. \qquad \text{§ 101}$$

Hence a circle with radius $r = IL$ is tangent to a, b, and c at L, M, and N, respectively. § 115

NOTE. The preceding proof involves also the proof of the **construction of the inscribed circle in any triangle.** Compare § 120.

125. Corollary 1. *A circle drawn from any point on the bisector of an angle, with a radius equal to the distance from that point to one side, is tangent to both sides of the angle.*

EXERCISES

1. How many circles can be drawn tangent to each of two given intersecting lines? What is the locus of their centers?

2. What is the locus of the centers of the circles of Ex. 1, in case the two given lines are parallel?

3. Show that a circle can be inscribed in a given quadrilateral if the bisectors of the four angles all meet in a single point.

4. Show that a circle can be inscribed in any square.

5. Show how to round off the vertices of any triangle by circular arcs. (See § 125.) This process is used in rounding off the corners of triangles that occur in a triangular-shaped piece of ground, so as to build a fence or a sidewalk or a building without sharp corners. (See also Ex. 5, p. 101.)

PART III. MEASUREMENT OF ANGLES

126. Numerical Measure. In order to measure any quantity, say a line of fixed length, we must first select the unit which we are to use. In the case of a fixed line, the customary unit would be either the inch, the foot, the centimeter, the yard, or any one of several others. In the case of an area, the unit might be a square inch or a square foot or an acre, or any one of several others.

Having once selected our unit, the process of measuring consists in obtaining some idea of the relative size of the given quantity as compared to that of the chosen unit. Thus, in the case of the fixed line, we lay a yardstick alongside of it, and read off by means of the scale provided for the purpose, a number which, at least with some degree of accuracy, tells us the relative size of the line in question to that of the inch.

Even though we cannot usually determine in this way the *exact* length of a line, owing to imperfections both in our instruments and our eyesight, still we suppose, and *it is indeed an* **axiom of measurement**, *that there always exists in every case just one number which does express exactly the length in terms of the unit.* This number is called the **numerical measure** of the given line, corresponding to the unit selected.

In general, the numerical measure of any quantity of any kind is a number, obtained as above, which expresses the relative size of the quantity to some unit of the same kind selected in advance.

127. Ratio. The numerical measure of one quantity divided by the numerical measure of a second quantity of the same kind, provided the same unit has been used in each case, is called the **ratio** of the first quantity to the second. Thus, the ratio of 6 feet to 8 feet is $6/8$ or $3/4$; again, the ratio of 6 inches to 1 yard is $6/36$ or $1/6$.

128. Commensurable and Incommensurable Quantities. Let AB and CD be two straight lines of different length and let it be supposed that a certain unit of length, as MN, is contained an exact (integral) number of times (that is, without any remainder) in AB. For example, let MN be contained 4 times in AB.

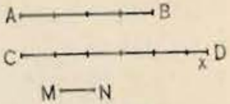


Fig. 90

Then MN may also be contained an exact number of times in the other line CD, but more often this will not be the case; ordinarily the unit MN will be contained in CD a certain exact number of times *plus* a remainder x, which will be less than MN. This is illustrated in Fig. 90, in which MN is contained 5 times in CD, with a remainder x.

Now, if we select a very small unit MN that is contained an exact number of times in AB as before, we obtain a very small remainder x when the same unit is applied to CD. However, it may happen that we can never take MN so small that the corresponding remainder x will turn out to be *exactly* zero. In this case the two lines AB and CD (Fig. 90) are said to be **incommensurable**.

Examples of incommensurable lines occur frequently in geometry. Thus, it will be seen later that in any isosceles right triangle a side and the hypotenuse constitute two incommensurable lines; that is, in Fig. 91, AB and BC are incommensurable. BC may be thought of as the diagonal of a square of which AB and AC are two sides.

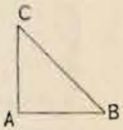


Fig. 91

Two other interesting incommensurable quantities are the diameter of any circle and the length of its circumference. These will be considered in detail in Chapter V.

If, on the other hand, it is possible to choose a unit which is contained an exact number of times in AB (Fig. 90), and also an exact number of times in CD, then the lines AB and CD are said to be **commensurable**.

129. Other Cases. Limits. Thus far we have spoken only of commensurable and incommensurable lines, but similar definitions apply to angles, arcs, or any other sorts of quantities. Thus, two angles are commensurable when a sufficiently small unit will be contained in each an exact number of times, and they are incommensurable when no unit which is contained an exact number of times in the one, will at the same time be contained an exact number of times in the other.

In any case, the remainder x can be made as nearly equal to zero as we please by taking the unit sufficiently small: we often say that x may be made to *approach zero*.

If a variable quantity approaches a fixed quantity as nearly as we please, the fixed quantity is called the **limit** of the variable one. Thus, in the preceding paragraph, we might say that the limit of x is zero.

EXERCISES

1. State accurately what is meant by two incommensurable arcs; two incommensurable areas.

2. Show that the remainder x of § 128 can always be made less than $\frac{1}{8}$ in.

3. Show that the remainder x of § 128 can be made smaller than one thousandth of an inch; one millionth of an inch. Hence show that any two lengths can be measured in terms of a common (small) unit, except for a remainder that is less than the human eye can see.

4. What is the smallest unit shown on your ruler? Can you see lengths less than this smallest division? If a line whose length you are measuring is not an exact number of these smallest units, how can you estimate its *exact* length?

5. Estimate the *exact* width of one line of type on this page.

6. The length of a material object will change on account of expansion due to changes in temperature. Show that a unit can be chosen so small that the remainder x (§ 128) will be less than the expansion due to a temperature change of one degree.

130. Theorem XI. *In the same circle, or in equal circles, two central angles have the same ratio as their intercepted arcs.*

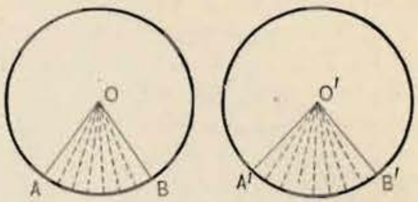


FIG. 92

Given the two equal circles O and O'; also let AOB be any central angle in O, and let $A'O'B'$ be any central angle in O'.

To prove that the ratio of $\angle AOB$ to $\angle A'O'B'$ is the same as that of arc AB to arc $A'B'$.

Proof. (*a*) *When arc AB and arc $A'B'$ are commensurable.*

In this case a unit of arc may be found (see § 128) which is contained an exact number of times in both the arc AB and the arc $A'B'$. (Compare Ex. 5, p. 91.) Let m be such a unit, and let us suppose that the number of times it is contained in arc AB is r, while the number of times it is contained in arc $A'B'$ is s. Then

$$(1)\qquad \frac{\text{arc } AB}{\text{arc } A'B'}=\frac{r}{s}.$$

Now divide the arc AB into its r divisions, each of length m, and through the points of division draw radii to the center O. Likewise, divide the arc $A'B'$ into its s divisions, each of the same length m, and draw radii through the points of division to the center O'. Then $\angle AOB$ is divided into r equal angles, while $\angle A'O'B'$ is divided into s equal angles of the same size (§ 104); therefore

$$(2)\qquad \frac{\angle AOB}{\angle A'O'B'}=\frac{r}{s}.$$

From (1) and (2) it follows that,

$$\frac{\angle AOB}{\angle A'O'B'}=\frac{\text{arc } AB}{\text{arc } A'B'}.$$

It would remain to prove Theorem XI when arc AC and arc $A'C'$ are *incommensurable*; but this proof is interesting only theoretically. Instead of giving a proof, we may assume as a **postulate** that *if any two geometric ratios are equal whenever their terms are commensurable, they are equal also when their terms are incommensurable.*

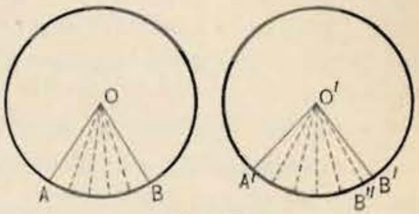


FIG. 93

The following proof may then be omitted at the discretion of the teacher:

Proof. (*b*) *When arc AB and arc $A'B'$ are incommensurable.*

In this case if we take any unit of arc m which is contained an exact number of times in the arc AB and apply it to the arc $A'B'$, there will remain after the last point of division a certain arc $B''B'$ less than m. § 128

But whatever the choice of m, we shall have

$$(3)\qquad \frac{\angle AOB}{\angle A'O'B''}=\frac{\text{arc } AB}{\text{arc } A'B''}. \qquad \text{Case } (a)$$

Now, as m is taken smaller and smaller, this equation (3) continues true at every step. At the same time, the individual members of the same equation are changing, but only to the extent that $\angle A'O'B''$ comes closer and closer to $\angle A'O'B'$, while arc $A'B''$ comes closer and closer to $A'B'$. Thus, by taking *m sufficiently small*, we can bring the first and second members of (3) as near as we please to the respective values

$$(4)\qquad \frac{\angle AOB}{\angle A'O'B'},\quad \frac{\text{arc } AB}{\text{arc } A'B'}.$$

These last ratios, then, differ by as little as we please from the *equal* ratios in (3); hence they themselves differ from *each other* by as little as we please. But this is the same as saying that these ratios (4) are actually equal, for, if they were *unequal* the difference between them could not by any method of reasoning be shown to be as small as we please, but would always remain greater than some definite amount; that is, in fact, what *unequal* means. Hence,

$$\frac{\angle AOB}{\angle A'O'B'}=\frac{\text{arc } AB}{\text{arc } A'B'}.$$

NOTE. Theorem XI shows that the number of angular units in a central angle is equal to the number of units of arc which the angle intercepts, if a unit of arc subtends a unit angle. Thus, the number of degrees in any central angle is the same as the number of degrees in the arc it intercepts. This fact is expressed by saying that

A central angle is measured by its intercepted arc.

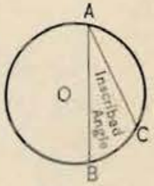


FIG. 94

131. Definition. An angle formed by the intersection of two chords in the circumference of a circle is called an **inscribed angle**; or, the angle is said to be **inscribed** in the circle.

132. Theorem XII. ***An inscribed angle is measured by one half of its intercepted arc.***

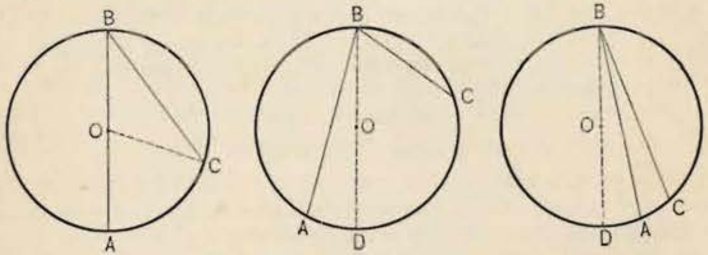


FIG. 95 (*a*) FIG. 95 (*b*) FIG. 95 (*c*)

Given the inscribed $\angle ABC$ intercepting $\widehat{AC}$ in the circle O.

To prove that $\angle ABC$ is measured by one half of the arc AC.

CASE 1. *When the center of the circle lies on one of the sides of the angle, as AB.* (Fig. 95 *a*.)

Proof. Draw OC; then	$OC = OB$;	Why?
hence	$\angle OBC = \angle OCB.$	Why?
But	$\angle OBC + \angle OCB = \angle AOC$;	Why?
therefore	$2\angle ABC = \angle AOC.$	Ax. 9
Since	$\angle AOC$ is measured by $\widehat{AC}$,	Note, § 130
it follows that	$\angle ABC$ is measured by $\frac{1}{2}$ AC.	Ax. 4

CASE 2. *When the center of the circle lies within the angle.*

Proof. Draw the diameter BD. (Fig. 95 *b*.)

Then	$\angle ABD$ is measured by $\frac{1}{2}\widehat{AD}$,	Case 1
and	$\angle DBC$ is measured by $\frac{1}{2}\widehat{DC}$;	Case 1
hence	$\angle ABD + \angle DBC$ is measured by $\frac{1}{2}(\widehat{AD} + \widehat{DC})$;	
that is,	$\angle ABC$ is measured by $\frac{1}{2}\widehat{AC}$.	

CASE 3. *When the center of the circle lies outside the angle.*

Proof. Draw the diameter BD. (Fig. 95 c.)

Then	$\angle DBC$ is measured by $\frac{1}{2}\widehat{DC}$,	Case 1
and	$\angle DBA$ is measured by $\frac{1}{2}\widehat{DA}$;	
hence	$\angle DBC - \angle DBA$ is measured by $\frac{1}{2}(\widehat{DC} - \widehat{DA})$;	
that is,	$\angle ABC$ is measured by $\frac{1}{2}\widehat{AC}$.	

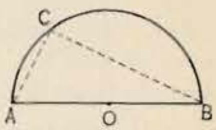


FIG. 96

133. Corollary 1. *Any angle inscribed in a semicircle is a right angle*; as the angle BCA in the figure.

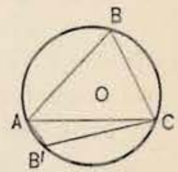


FIG. 97

134. Corollary 2. *Any angle inscribed in a segment* (see § 131) *greater than a semicircle is acute, while any angle inscribed in a segment less than a semicircle is obtuse.*

Thus, in the figure, ABC is acute, while $AB'C$ is obtuse.

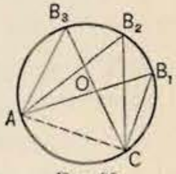


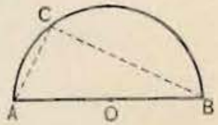
FIG. 98

135. Corollary 3. *All angles inscribed in the same segment are equal.*

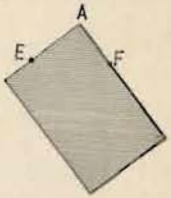
Thus, in the figure, the angles AB_1C, AB_2C, AB_3C, are all equal.

EXERCISES

1. A thin elastic band is stretched along the diameter AB of a circle and then pinned firmly to the circumference at the two points A and B. If it be now stretched aside by means of a pencil point so that it takes the position indicated by the dotted line; that is, so that a third point C of the band lies on the circumference, what can be said of the angle ACB? As the pencil is allowed to move about the circumference, the band meanwhile sliding over the pencil point, how does the angle ACB change?



2. A corner of a piece of cardboard (of the usual rectangular shape) is pressed tightly against two pins E and F stuck into a board below. The cardboard is now turned in all possible positions, keeping it flat against the board. What is the locus of the point A?



3. Prove by means of § 132 that the sum of the three angles of a triangle is two right angles.

4. If a circumference be divided into four equal arcs (quadrànts), show that the chords which join the extremities form a square.

5. Show by § 132 that any parallelogram inscribed in a circle is a rectangle.

6. What is the locus of all the vertices of right-angled triangles erected on a common hypotenuse?

7. Prove that the opposite angles of any inscribed quadrilateral are supplementary.

8. Prove that any equilateral polygon inscribed in a circle is also equiangular.

136. Theorem XIII. *An angle formed by two chords intersecting within a circle is measured by one half of the sum of the intercepted arcs.*

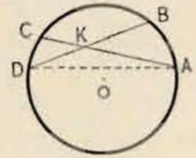


FIG. 99

Outline of proof. Draw AD.

Then	$\angle AKB = \angle D + \angle A.$	§ 64
But	$\angle D$ is measured by $\frac{1}{2}\widehat{AB}$	§ 132
and	$\angle A$ is measured by $\frac{1}{2}\widehat{CD}$;	§ 132
hence	$\angle AKB$ is measured by $\frac{1}{2}(\widehat{AB} + \widehat{CD})$.	Why?

137. Theorem XIV. *An angle formed by a tangent and a chord drawn through the point of tangency is measured by one half of the intercepted arc.*

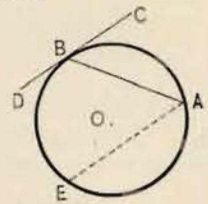


FIG. 100

Outline of proof. Draw $AE \parallel CD$.

Then	$\widehat{AB} = \widehat{BE}$	§ 119
and	$\angle ABC = \angle A.$	§ 54
But	$\angle A$ is measured by $\frac{1}{2}\widehat{BE}$;	Why?
hence	$\angle ABC$ is measured by $\frac{1}{2}\widehat{AB}$.	

138. Theorem XV. *An angle formed by two secants, or by a tangent and a secant, or by two tangents that meet outside a circle is measured by one half the difference of the intercepted arcs.*

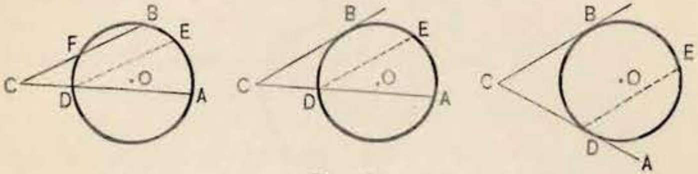


FIG. 101

[The proofs are left to the student. Draw DE parallel to BC and note how the angle ADE, which is equal to C, is measured.]

EXERCISES

1. Prove Theorem XIII by drawing a line through A (Fig. 99) parallel to BD.

2. If, in Fig. 99, the arc BC contains 130° and the arc $ABCD$ contains 170°, how many degrees are there in the angle AKB? *Ans.* 20°.

3. Two angles of an inscribed triangle are 70° and 91°. Find in degrees the arcs subtended by each of the sides.

4. A chord that divides a circumference into arcs one of which contains 75° is met at one extremity by a tangent. At what (acute) angle does the meeting take place?

5. A chord is met at one extremity by a tangent, making with it an angle of 61°. Into what arcs does the chord divide the circumference?

6. If a tangent is drawn at the vertex of an inscribed square, how many degrees are there in the angle included between the tangent and a side of the square? Answer the same question for an inscribed equilateral triangle.

PART IV. CONSTRUCTION PROBLEMS

139. Problem 1. *Through a given point to draw a tangent to a circle.*

CASE 1. *When the point is on the circumference.*

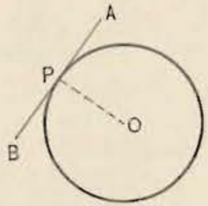


FIG. 102

Given the ⊙ O and the point P on the circumference.

Required to draw a tangent to the ⊙ O through P.

Construction. Draw the radius OP. At P draw $AB \perp OP$. Then AB is the required tangent. Why?

CASE 2. *When the given point is not on the circumference.*

Outline of construction. On OP as diameter draw a circle cutting the given circle at A and B.

Pass a line through the points P, A, and another line through the points P, B.

Either of these lines is a tangent such as desired. Why? In answering, note that $\angle PAO$ and $\angle PBO$ are right angles and apply § 115.

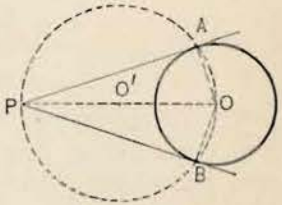


FIG. 103

EXERCISES

1. Show how to construct a tangent to a given circle parallel to a given line.

2. Show how to draw a tangent to a given circle perpendicular to a given line.

140. Problem 2. *To circumscribe a circle about a given triangle.* (See § 120. The student should carefully state the construction and the proof.)

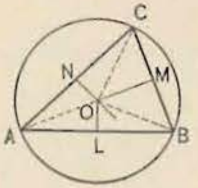


FIG. 104

141. Problem 3. *To inscribe a circle in a given triangle.* (See § 124. State carefully the construction and its proof.)

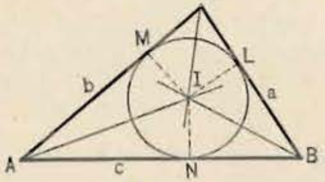


FIG. 105

EXERCISES

1. Can a circle always be drawn tangent to three lines, no two of which are parallel? If so, how?

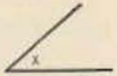
2. Construct an equilateral triangle each of whose sides equals 2 inches, and then construct its inscribed circle. Prove that in this case the center of the circle lies at the intersection of the three altitudes of the triangle. Can the same statement be made for any other triangle?

3. In Fig. 105, would the lines BI and AI (extended) ever pass through the points of contact M, L between the inscribed circle and the sides AC and BC, respectively? If so, when would they, and why?

142. Problem 4. *On a given straight line to construct a segment of a circle that shall contain a given angle.*

Given line AB and $\angle x$.

Required to construct on AB a segment of a circle that shall contain the $\angle x$.



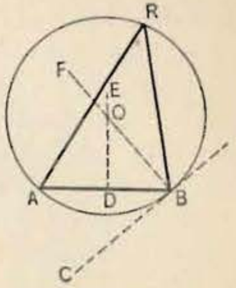


FIG. 106

Construction. Construct $\angle ABC = \angle x$.

Draw $ED \perp AB$ at its middle point D.

Draw $FB \perp CB$ at B and meeting ED in O.

Draw a circle with O as center and OB as radius.

Then ARB is the segment required.

Proof. $\angle ARB = \angle ABC = \angle x$. Why?

EXERCISES

1. On a line 2 inches long construct a segment of a circle that shall contain an angle of 60°. Do the same for 30°; 135°.

2. Construct the locus of the vertices of all triangles having a common base 2 inches long and a common vertex angle of 30°.

3. In Ex. 2, p. 114, what can be said of the locus when the angle A is 60° instead of 90°?

4. Construct the triangle whose base is 2 inches, whose altitude is 3 inches, and whose vertex angle is 30°.

[HINT. First, draw the segment of a circle on a base of 2 inches, and such that 30° is contained in it. Now, draw a parallel to the same base at a distance of 3 inches above it. Where this parallel cuts the circle bounding the segment, will be the vertex of the triangle desired. Why? The desired triangle is now easily drawn.]

5. How many solutions will there be to any problem similar to Ex. 4? May there be only one solution? May there be no solution? Describe all possible cases.

MISCELLANEOUS EXERCISES. CHAPTER II

1. Prove that the construction described in § 5 for drawing a perpendicular to a straight line at a point within it is correct.

2. Prove that the constructions of §§ 6 and 8 are correct.

3. Through a given point in a circle construct the chord that is bisected by that point.

4. If a right triangle be inscribed in a circle, show that its hypotenuse will be a diameter.

5. Prove that an exterior angle of an inscribed quadrilateral equals the opposite interior angle.

6. Prove that the angle between two tangents to a circle is double the angle between the chord joining the points of contact and the radius to a point of contact.

7. Construct a circle such that it shall pass through two given points and shall have its center on a given line.

8. If four points A, B, C, D lie on a circle, and if $\widehat{AB}=\widehat{CD}$ and $\widehat{AD}=\widehat{BC}$, show that the lines AC and BD meet at the center.

9. Prove that the shortest line from a point to a circumference is along the radius through the point, extended if necessary, (*a*) when the point is outside the circle; (*b*) when the point is inside the circle.

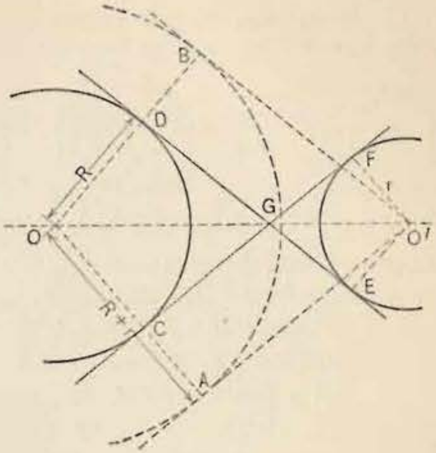
10. Prove that the angle between two tangents to a circle is the supplement of the angle between the radii drawn to the points of tangency.

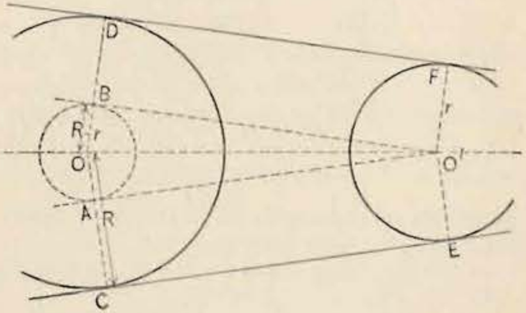
11. If the sides of an inscribed angle are parallel to the sides of a central angle, how do their arcs compare?

12. Show that two tangents to the same circle are parallel if and only if their points of tangency lie at opposite ends of the same diameter.

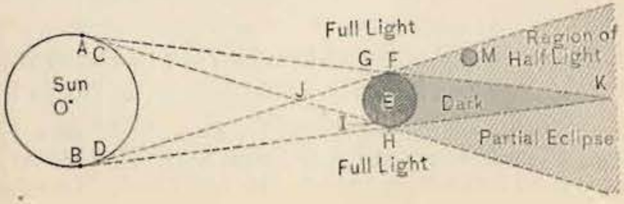
13. Show that a circle whose diameter is one side of any given triangle passes through the feet of the altitudes drawn to each of the other sides.

14. Following the suggestions of the adjoining figures, show how to construct four common tangents to two given circles if the circles do not intersect.

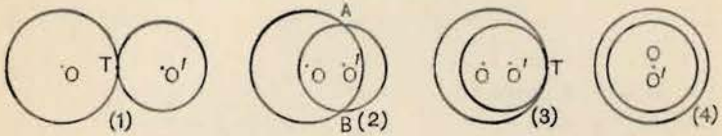




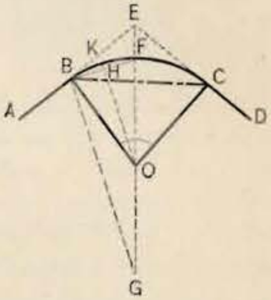
NOTE. The shadows thrown by a round body illuminated by a round source of light (*e.g.* earth and sun) illustrate this exercise.



15. How many common tangents can be drawn to two given circles in each of the cases illustrated in the following figures?



16. A common railroad turn is made by an arc of a circle tangent to two straight portions (Ex. 5, p. 101). In the adjoining figure, if $OG = OF$, and $OK \perp BF$, show that:



(1) $\angle OGB = \angle FOB/2 = \angle COB/4$.

(2) OK is parallel to BG.

(3) $\widehat{BK} = \widehat{BC}/4$,

and $\angle OGB = \angle FOH$.

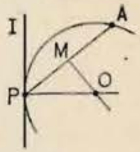
(4) $\angle EBF = \angle BOK$,

and $\angle EBC = \angle BOF$.

17. If in Ex. 16, $\angle COB = 60°$, what is the length of the chord BC? Find approximately, from an accurate figure, or from the table of chords, p. iii, the length of the chord BF. Prove that $BF = FC > BC/2$.

NOTE. In practice the entire curve BC can be (and frequently is) laid off by means of a surveying instrument located at B, by knowing accurately the lengths of a number of such chords through B, and their directions.

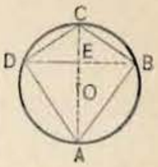
18. Construct a circle through a given point tangent to a given line at a given point.



19. Construct the triangle whose base is 2 inches, whose median to the same base is 3 inches, and whose vertical angle is 30°.

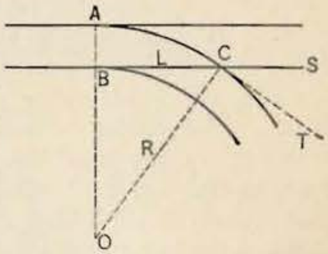
[HINT. The vertex must lie at the intersection of two circles, one bounding the segment mentioned in the Hint to Ex. 4, p. 119, and the other described about the middle point of the base with a radius of 3 inches.]

20. Show that a quadrilateral can be inscribed in a circle if the sum of one pair of its opposite angles is 180°.



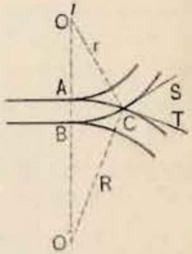
21. Show that a *kite* (see Ex. 30, p. 87) that is inscribed in a circle has two right angles; hence show that one diagonal is a diameter.

22. A simple turnout off of a straight track on street railways is of the form of a circular arc, as shown in the figure. The length $R = OA = OC =$ the radius of the outer rail of the circular track, is called the *radius of the turnout*. The distance $L = BC$ from the point of the switch B to the crossing C is called the *lead*, and the angle TCS between the tangent to the circular track at C and the straight track, is called the *angle of crossing*.



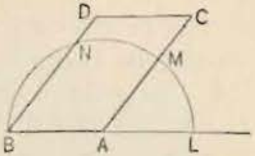
Show that if R is given (as it usually is), the angle SCT of crossing (which must be known accurately for the purpose of ordering the proper rail cross) can be found from the figure by measuring the angle COA.

23. A double opposite turnout from the end of a straight track is composed of two circular portions, the radii of the outer rails being given lengths r and R, respectively, and these outer rails being tangent to the straight track from which they start. Show that the angle of crossing SCT is the sum of the two angles COA and $CO'B$.

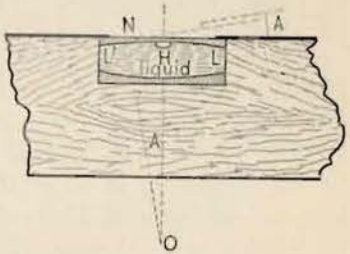


24. Let D be any point on the side AC of a right triangle ABC, and let E be the intersection of the hypotenuse AB with a circle whose diameter is AD. Show that the quadrilateral $DCBE$ is inscribed in the circle whose diameter is DB.

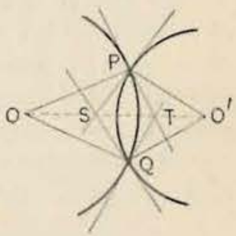
25. If from one corner A of a parallelogram $ABCD$ as center, a circle of radius AB is drawn, meeting AB produced at L, and the sides AC and BD at M and N, respectively, show that $\widehat{LM} = \widehat{MN}$.



26. The usual form of carpenter's level contains a glass tube filled with liquid, except for a small bubble of air. The inner surface of the tube is cut in the form of a circular arc LHL', whose center is at O; and it is set into a wooden or metal frame whose top edge is parallel to the tangent at H. Show that when the level is tilted through the angle A, the bubble moves through an arc HN which subtends the same angle A at the center O. Will the level be more sensitive if the radius R $(= OH)$ is large or small?



27. If two circles intersect, show that the obtuse angle between the two tangents to the two circles at either point of intersection is the same as the obtuse angle between the two radii drawn to that point. Hence show that the angle between the two tangents at one point of intersection is the same as that between the two tangents at the other point.



28. Prove that the sum of two opposite sides of a circumscribed quadrilateral equals the sum of the other two sides.

29. Show how to construct the common tangents to the pairs of circles represented in (1), (2), (3), Ex. 15, p. 122.

30. Construct a rectangle having given a side and the diagonal.

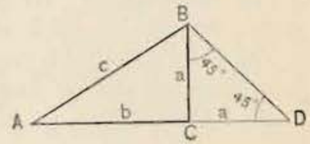
31. Construct an isosceles triangle having given the base and the length of the altitude from one end of the base to one of the equal sides.

32. Construct a circle whose center lies on one side of a given triangle, and which is tangent to each of the other two sides.

33. Construct a triangle, having given the base and the altitudes to the other two sides.

34. Construct a triangle, having given the base, the altitude to the base, and a second altitude.

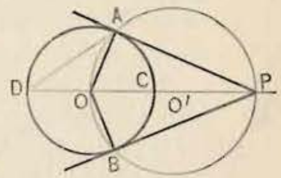
35. Construct a right triangle, from the suggestion contained in the figure, when the hypotenuse AB and the *sum* AD of the two sides are given.



36. Show that the bisectors of opposite angles of a parallelogram are parallel or coincide. Hence show that the only kind of parallelograms in which a circle can be inscribed is a rhombus.

37. Show that the figure formed by two tangents PA and PB to a circle O and the radii OA and OB forms a kite which is inscribed in a circle O' whose diameter is OP.

Let OO' meet the circle O at C and D. Show that $\angle COA$ is measured by half the arc AP; and $\angle CDA$ by one fourth the arc AP.



38. In the figure for Ex. 37, draw the radius $O'A$. Show that any change in $\angle CDA$ causes four times as much change in $\angle PO'A$.

39. Show what becomes of the figure of Ex. 37 if the angle COA is equal to $90°$.

CHAPTER III

PROPORTION SIMILARITY

PART I. GENERAL THEOREMS ON PROPORTION

143. Definitions. An equality of two ratios is called a **proportion**.

Thus, the ratio $2/3$ being equal to the ratio $4/6$ gives us the proportion $2/3 = 4/6$. More generally, if the ratio a/b is equal to the ratio c/d, then the equality $a/b = c/d$ is a proportion.

The form $\frac{a}{b} = \frac{c}{d}$ is identical with $a/b = c/d$, since a/b is only a convenient way of printing a fraction.

Besides writing a proportion in the form $a/b = c/d$, either of the following forms may also be used, $a : b = c : d$, or $a : b :: c : d$. In any case, the proportion is read "*a is to b as c is to d.*"

In the proportion $a : b = c : d$, the numbers a, b, c, and d are the *terms*. The first and fourth of these (a and d) are called the *extremes*, while the second and third (b and c) are called the *means*. Again, the first and third (a and c) taken together are called the *antecedents*, while the second and fourth taken together are called the *consequents*.

A series of equal ratios in the form $a/b = c/d = e/f = \cdots$ is called a *continued proportion*. Thus, $2/3 = 4/6 = 8/12 = \cdots$.

EXERCISES

1. Using the language of proportion, read the equation $\frac{5}{6} = \frac{10}{12}$. What are the extremes here, what the means; which the antecedents and which the consequents?

2. Form and read off such proportions as you can make out of the following four quantities: 2 in., 8 in., 4 in., 16 in. Do the same for 1 in., 3 in., 1 ft., 1 yd.

3. What number bears the same ratio to 2 as 8 does to 3?

[HINT. In this and in all similar questions, let x represent the desired number and form an equation. Thus, we here have $x/2 = 8/3$. Now solve this equation for x.]

4. Find two numbers such that their ratio is as $3:5$ and whose sum is 4.

[HINT. Let x and y represent the unknown numbers and form equations.]

144. General Theorems on Proportion.

Theorem A. *In any proportion, the product of the extremes is equal to the product of the means.*

Given $a/b = c/d$, **to prove** that $ad = bc$.

Proof. Clear the given equation of fractions by multiplying both its members by bd. The result is the desired equation $ad = bc$.

Corollary 1. *If the two antecedents of a proportion are equal, the consequents are also equal.*

Theorem B. *If the product of two numbers is equal to the product of two other numbers, either pair may be made the means of a proportion in which the other two are taken as the extremes.*

Given $ad = bc$, **to prove** that $a/b = c/d$.

Proof. Divide both members of the given equation by bd. In order to prove that $b/a = d/c$, write the given equation in the form $bc = ad$ and then divide both members by ac.

Theorem C. *If four quantities are in proportion, they are in proportion by* **inversion**; that is, the second term is to the first as the fourth is to the third.

Given $a/b = c/d$, **to prove** that $b/a = d/c$.

Proof. $ad = bc$ by Theorem A; hence $b/a = d/c$ by Theorem B.

Theorem D. *If four quantities are in proportion, they are in proportion by* **alternation**; that is, the first term is to the third as the second is to the fourth.

[The proof is left to the student. First use Theorem A.]

Theorem E. *If four quantities are in proportion, they are in proportion by* **composition**; that is, the sum of the first two terms is to the second term as the sum of the last two terms is to the last term.

Given $a/b = c/d$, **to prove** that $(a+b)/b = (c+d)/d$.

Proof. Adding 1 to each side of the given equation, we have

$$\frac{a}{b}+1=\frac{c}{d}+1, \text{ or } \frac{a+b}{b}=\frac{c+d}{d}. \qquad \text{Why ?}$$

[Show in a similar way that $(a+b)/a = (c+d)/c$.]

Theorem F. *If four quantities are in proportion, they are in proportion by* **division**; that is, the difference between the first two terms is to the second term as the difference between the last two terms is to the last term.

Given $a/b = c/d$, **to prove** that $(a-b)/b = (c-d)/d$.

Outline of proof. Make use of the fact that $a/b - 1 = c/d - 1$.

[Show in a similar way that $(a-b)/a = (c-d)/c$.]

Theorem G. *If four quantities are in proportion, they are in proportion by* **composition and division**; that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.

Given $a/b = c/d$, **to prove** that $(a+b)/(a-b) = (c+d)/(c-d)$.

Proof. We have

$$\frac{a+b}{b}=\frac{c+d}{d}, \text{ and } \frac{a-b}{b}=\frac{c-d}{d}. \qquad \text{Th. E, F}$$

Therefore $(a+b)/(a-b) = (c+d)/(c-d)$. Ax. 4

Theorem H. *In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Given $a/b = c/d = e/f = \cdots$, **to prove** that

$$\frac{a+c+e+\cdots}{b+d+f+\cdots}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\cdots.$$

Proof. Let k be the value of any one of the equal ratios

$$\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \cdots,$$

so that

$$k=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\cdots.$$

Then

$$a=kb,\ c=kd,\ e=kf,\ \cdots; \qquad \text{Why ?}$$

hence

$$\frac{a+c+e+\cdots}{b+d+f+\cdots}=\frac{k(b+d+f+\cdots)}{b+d+f+\cdots}=k, \qquad \text{Ax. 9}$$

or,

$$\frac{a+c+e+\cdots}{b+d+f+\cdots}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\cdots.$$

EXERCISES

1. Given the proportion $1/2 = 3/6$. Write down the various proportions which come from this by (1) inversion, (2) alternation, (3) composition, (4) division, (5) composition and division. Note that each one thus obtained is a *true* proportion.

2. If $a/b = c/d$, prove that $ma/mb = c/d$, where m represents any number.

3. If $a/b = c/d$, prove that $2a/3b = 2c/3d$; also that $(a+3b)/b = (c+3d)/d$.

4. If $a/b = b/c$, prove that $b^2 = ac$; also that $a/c = a^2/b^2$.

5. If $b^2 = ac$, prove that $a/b = b/c$.

6. If $a/b = b/c$, prove that $c = b^2/a$.

PART II. PROPORTIONAL LINE-SEGMENTS

145. Theorem I. ***A line parallel to the base of a triangle divides the other sides proportionally.***

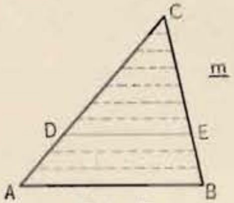


FIG. 107

Given the $\triangle ABC$ and the line $DE \parallel$ the base AB.

To prove that $\quad CD/DA = CE/EB$.

Proof. (*a*) *When CD and DA are commensurable.* (§ 128)

In this case a unit length may be found which is contained an exact number of times in both CD and DA. Let m be such a unit and let us suppose that this unit is contained r times in CD and s times in DA.

Then,

$$\frac{CD}{DA} = \frac{r}{s}. \tag{1}$$

Now divide CD into its r divisions, each of length m, and through the points of division draw lines parallel to AB. Do the same with DA.

These parallels will cut CE into r equal parts and EB into s equal parts. (§§ 91–92.) Therefore

$$\frac{CE}{EB} = \frac{r}{s}. \tag{2}$$

Comparing (1) and (2) we obtain, as was to be proved,

$$\frac{CD}{DA} = \frac{CE}{EB}.$$

NOTE. As in § 130, this essentially completes the proof; and the theorem holds also in the case CD and DA are incommensurable, by the postulate of § 130.

The following argument establishes the truth of that postulate in the case of the present theorem; it may be omitted at the discretion of the teacher.

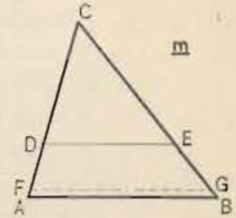


FIG. 108

Proof. (*b*) *When CD and DA are incommensurable.*

In this case if we take any unit length m which is contained an exact number of times in CD and apply it to DA, there will remain after the last point of division a certain length AF which will be less than m.

But, whatever the choice of m, we shall have, by case (*a*), $CD/DF = CE/EG$.

Now, as m is taken smaller and smaller, this equation holds true at every step. At the same time the individual members of the same equation are changing, but only so that DF comes closer to DA, while EG comes closer to EB. By taking m *sufficiently small* we can thus bring CD/DF and CE/EG as near as we please to the respective values CD/DA and CE/EB. These last ratios, since they differ by as little as we please from the preceding *equal* ratios, must differ from *each other* by as little as we please. This is the same as saying that they are actually equal; for, if unequal, their difference could not by any method of reasoning be made as small as we please, since they are fixed quantities.

Thus we must have, as was to be proved, $CD/DA = CE/EB$.

146. Corollary 1. *If a line is drawn parallel to the base of a triangle, either side is to one of its segments as the other side is to its corresponding segment.*

Given the $\triangle ABC$ and $DE \parallel AB$. (See Fig. 107.)

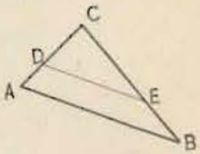
To prove (1) that $CA/CD = CB/CE$

and (2) that $CA/DA = CB/EB$.

[**Outline of proof.** Having shown in Theorem I that $CD/DA = CE/EB$, we obtain by composition (E, § 144) $(CD + DA)/DA = (CE + EB)/EB$, from which (1) follows. To prove (2), taking the proportion $CD/DA = CE/EB$ by inversion (C, § 144) we have $DA/CD = EB/CE$. Now take this proportion by composition.]

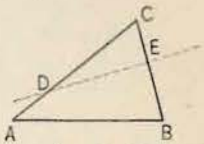
EXERCISES

1. In the adjacent figure DE is parallel to AB. If $CD = 2$ in., $CA = 3$ in., and $CE = 3\frac{1}{2}$ in., what is the length of EB? *Ans.* $1\frac{3}{4}$ in.

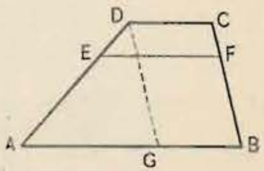


2. If, in the figure of Ex. 1, we have $CA = 8$ ft., $CB = 12$ ft. 3 in., and $CE = 9\frac{1}{3}$ ft., what is the length of DA?

3. A thread DE is stretched in any direction across a triangle ABC cutting two of its sides in the points D, E. The thread is now moved up and down over the surface of the triangle, but always in such a way as to be parallel to its original direction. How does the ratio CD/CE change during the motion? Why?

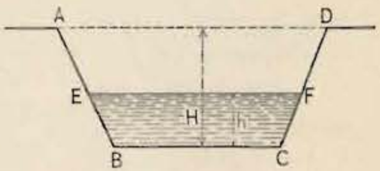


4. Prove that any line drawn parallel to the base of a trapezoid cuts the other two sides proportionally.



5. If a common exterior tangent to two circles (see figures for Ex. 14, p. 121) meets the line of centers OO' in a point G, show that the lengths along this tangent from G to the two circles are in the ratio $OG/O'G$.

6. A channel for water has a cross-section of the form of a trapezoid $ABCD$. If BC and AD are horizontal, show that the wetted portions of the side walls BE and CF are proportional to the dry portions AE and DF.



7. What other proportions can you write down for Ex. 6?

8. In Ex. 6, if $AB = 15$ ft., $CD = 12$ ft., and the vertical height H from BC to AD is 10 ft., find the lengths of the dry portions of the side walls when the water is 4 ft. deep.

147. Theorem II. (*Converse of Theorem I.*) *If a line divides two sides of a triangle proportionally, it is parallel to the third side.*

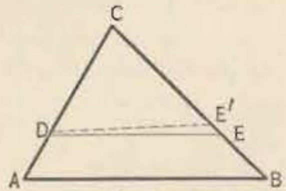


FIG. 109

Given the $\triangle ABC$ and the line DE dividing the sides proportionally; that is, such that $CD/DA = CE/EB$.

To prove that DE is parallel to AB.

Proof. Suppose that DE is not $\parallel AB$. Then draw $DE' \parallel AB$. We should now have

(1) $CA/CD = CB/CE'$. § 146

Also, since by hypothesis

$$CD/DA = CE/EB,$$

it follows that

(2) $CA/CD = CB/CE$. Th. E, § 144

From equations (1) and (2) we should have

$CE' = CE$; Th. A, Cor. 1, § 144

whence DE' would coincide with DE. Post. 1

This is impossible unless $DE \parallel AB$. Why?

Therefore, $DE \parallel AB$.

148. Corollary 1. *If a line cuts two sides of a triangle in such a way that either side is to one of its segments as the other side is to its corresponding segment, then the line is parallel to the third side.*

[HINT. Show that under the given conditions the hypothesis of § 147 is satisfied. Compare with § 89 and § 146.]

149. Theorem III. *The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the sides of the angle.*

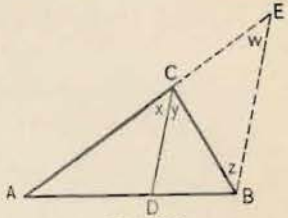


FIG. 110

Given the $\triangle ABC$ and the bisector CD of $\angle C$.

To prove that $AD/DB = AC/BC$.

Proof. Draw $BE \parallel CD$ and meeting AC prolonged at E.

Then $AD/DB = AC/CE$. Why?

Moreover, $\angle x = \angle y$, $\angle x = \angle w$, $\angle y = \angle z$. Why?

Therefore, $\angle z = \angle w$; whence $CE = BC$; Why?

hence $AD/DB = AC/BC$. Ax. 9

EXERCISES

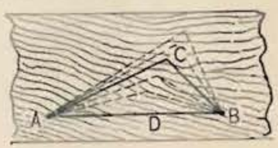
1. In the triangle ABC of Fig. 110, $CA = 11$ in., $CB = 8$ in., and $AB = 14$ in. What are the lengths of the segments AD and DB into which the bisector CD of the angle C divides the base AB? *Ans.* $AD = 8\frac{2}{19}$ in., $DB = 5\frac{17}{19}$ in.

2. Prove by means of § 147 that the line joining the middle points of two sides of a triangle is parallel to the third side and equal to half of it. (Compare with § 89.)

3. State and prove the converse of Theorem III.

[HINT. Prolong AC to E, making $CE = BC$. Draw EB. (Fig. 110.) Prove $AC/AD = CE/DB$, and hence $CD \parallel EB$. Then $\angle x = \angle w$ and $\angle y = \angle z$. But $\angle w = \angle z$ (since $CB = CE$). Therefore $\angle x = \angle y$.]

4. An ordinary closed rubber band is stretched out and held against a board by means of two pins A and B. One of the halves of the band is stretched by means of a pencil point into the form of an elastic triangle ABC, with the pencil point at C. How must the pencil point move in order that the ratio CA/CB shall remain *constant?*



150. Theorem IV. *If a series of parallels be cut by two lines, the corresponding segments are proportional.*

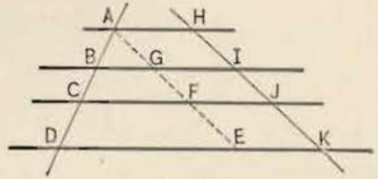


FIG. 111

Given the parallels AH, BI, CJ, etc., cut by the lines AD and HK.

To prove that $AB/HI = BC/IJ = CD/JK$.

Proof. Draw $AE \parallel HK$.

Now $AC/AF = AB/AG = BC/GF$ Why?

and $AC/AF = CD/FE$. Why?

Therefore $AB/AG = BC/GF = CD/FE$. Why?

But $AG = HI$, $GF = IJ$, $FE = JK$; Why?

hence $AB/HI = BC/IJ = CD/JK$. Ax. 9

NOTE. All work on proportional segments is based on §§ 145, 150. See §§ 149, 151, 153, etc. Compare also §§ 91–94.

EXERCISES

1. Prove Theorem IV when the two lines AD, HK (Fig. 111) intersect at some point between the parallels AH, DK.

2. Prove the fact stated in Cor. 1, § 92 by means of § 150.

3. If the three segments cut out of one transversal by four parallel lines are 2 in., 3 in., 4 in., respectively, what can be inferred concerning the perpendicular distances between the parallels? If the distance between the first pair is 1 in., what are the distances between the others?

151. Problem 1. *To divide a given line into parts proportional to any number of given lines.*

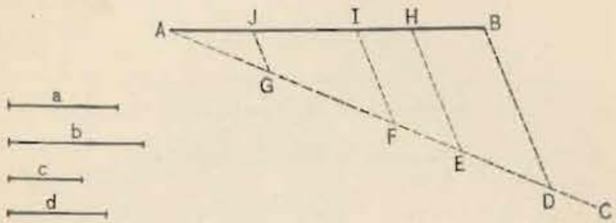


FIG. 112

Given the line AB.

Required to divide AB into parts which shall be proportional to the given lengths a, b, c, d.

Construction. Draw AC making any angle with AB.

Measure off $AG = a$, $GF = b$, $FE = c$, $ED = d$.

Join the last point D thus determined with B, and through E, F, and G draw parallels to BD cutting AB in H, I, and J.

Then the segments AJ, JI, IH, HB, divide AB as required.

Proof. $AJ/AG = JI/GF = IH/FE = HB/ED$; § 150

hence $AJ/a = JI/b = IH/c = HB/d$.

NOTE. This problem includes as a special case, *the division of a given line into any number of equal parts;* for if we take $a = b = c = d$, we have $AG = GF = FE = ED$.

EXERCISES

1. Divide a line 10 in. long into 8 equal parts by Note, § 151. Show how to do the same exercise by § 8.

2. Divide a line 10 in. long into 7 equal parts by § 151. Can this exercise be done by § 8?

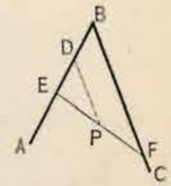
3. State when § 8 can be used in place of Note, § 151 to divide a given line into a number of equal parts.

4. How could the page of a notebook be ruled (accurately) into three equal columns?

5. Divide a given line into parts which are to each other as $1:2:3$.

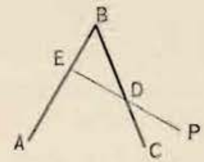
6. Draw a line AB and divide it into fifths; thereby find the point on AB one fifth the distance from A to B. Find a point $2/5$ of AB from A.

7. Given an angle ABC and a point P anywhere within it. To construct a line through P such that the portion of the line lying within the angle shall be bisected by P.



[HINT. Through P draw $PD \parallel BC$. Then lay off $DE = DB$.]

8. Through a point P without an angle ABC to draw PE so that $PD = DE$.



9. State and prove a theorem concerning the segments into which the non-parallel sides of a trapezoid are divided by a series of parallels to the bases.

10. Layers of rock in the earth are nearly parallel, but they are not usually level (horizontal). Show that the ratios of the thicknesses of layers of different kinds of rock can be found by measuring the thicknesses of the cuts made by a vertical mine-shaft, or a deep well.

152. Definitions. In the proportion $a/b = c/d$, d is called the **fourth proportional** to a, b, and c. Thus, 6 is the fourth proportional to 1, 2, and 3. If $b = c$, so that $a/b = b/d$, d is called the **third proportional** to a and b.

153. Problem 2. *To find the fourth proportional to three given lines.*

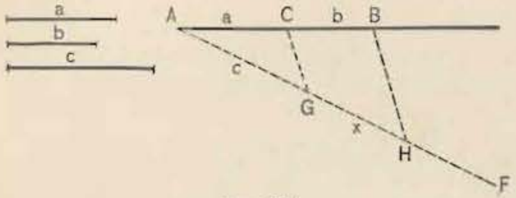


FIG. 113

Given the three lines a, b, and c.

Required to construct their fourth proportional; that is, to find a line x such that we shall have $a/b = c/x$.

Construction. Draw a line AB of indefinite extent and from A lay off AC equal to a and CB equal to b.

Draw a second line AF of indefinite extent, making any convenient angle at A, and along AF lay off AG equal to c.

Join CG. Draw $BH \parallel CG$ and cutting AF at H.

Then GH is the required line x.

[The proof is left to the student. Use § 145.]

EXERCISES

1. Find a fourth proportional to three lines 2 in., 4 in., and 5 in. long, respectively. Find a fourth proportional to three lines 2 in., 5 in., and 4 in. long, respectively.

2. Show how to construct a third proportional to two given lines.

3. To enlarge a line segment l in the ratio 2/3 is to find a new line segment x such that $2/3 = l/x$. Show how to do this geometrically.

4. Draw a triangle; then enlarge its sides in the ratio 3/5, using geometric constructions only. See § 21, p. 19.

PART III. SIMILAR TRIANGLES AND POLYGONS

154. Definitions. That a figure may be *enlarged* or *reduced* in size without essentially changing its appearance otherwise is a part of common knowledge. Thus we drew such enlarged and reduced figures in § 21. Throughout this book rather small figures are printed; these the student almost always enlarges in drawing on paper or at the board.

In general, nearly all drawings represent objects of a size different from that of the drawing; thus house plans are usually drawn (§ 21) on the scale of $\frac{1}{4}$ inch to one foot.

Enlargements, or reductions, are made by increasing, or decreasing, all lengths in a figure in the same ratio throughout, so that all lengths in the new drawing are *proportional* to the corresponding lengths in the original figure.

Two triangles which have their corresponding sides proportional are called **similar**. Fig. 114 represents two similar triangles, each side of the one being twice that of the other.

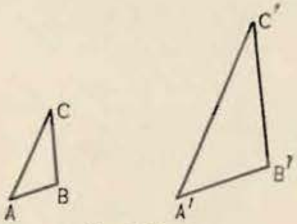


FIG. 114

The fact that two triangles ABC and $A'B'C'$ are similar is expressed by the relation $ABC \sim A'B'C'$.

In general, if one figure is obtained from another by enlargement or reduction, that is, if *all* lengths that can be drawn in one are proportional to the corresponding lengths in the other, the two figures are said to be **similar**.

Two triangles are said to be **mutually equiangular** when their corresponding angles are equal.

155. Theorem V. ***If two triangles are mutually equiangular, they are similar.***

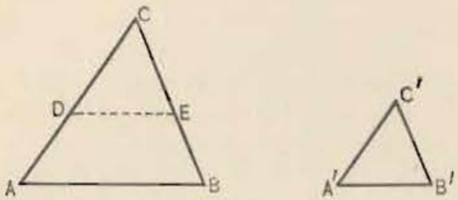


FIG. 115

Given the mutually equiangular $\triangle$ ABC, $A'B'C'$.

To prove that $\triangle ABC \sim \triangle A'B'C'$.

Proof. On CA lay off $CD = C'A'$ and on CB lay off $CE = C'B'$. Draw DE. Then, in the $\triangle$ DEC and $A'B'C'$ we have

have	$CD = C'A'$, $CE = C'B'$,	Const.
and	$\angle C = \angle C'$.	Given
Therefore	$\triangle DEC \cong \triangle A'B'C'$.	Why?
Whence	$\angle CDE = \angle C'A'B'$.	
But	$\angle A = \angle C'A'B'$.	Given
Hence	$\angle CDE = \angle A$.	Why?
It follows that	$DE \parallel AB$;	Why?
hence also,	$\frac{CD}{CA} = \frac{CE}{CB}$ or $\frac{C'A'}{CA} = \frac{C'B'}{CB}$.	§ 146

In like manner it can be shown that $\frac{C'A'}{CA} = \frac{A'B'}{AB}$.

Therefore $\triangle ABC \sim \triangle A'B'C'$. § 154

156. Corollary 1. *Two triangles are similar if two angles of the one are equal respectively to two angles of the other.*

157. Corollary 2. *Two right triangles are similar if an acute angle of the one is equal to an acute angle of the other.*

158. Theorem VI. *If two triangles are similar, they are mutually equiangular.* (*Converse of Theorem V.*)

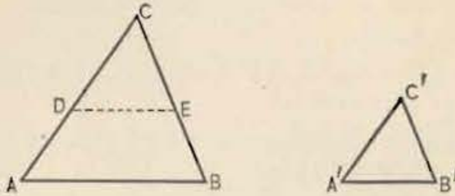


FIG. 116

Given the two similar $\triangle$ ABC and $A'B'C'$; that is, two $\triangle$ such that $\frac{CA}{C'A'} = \frac{CB}{C'B'} = \frac{AB}{A'B'}$.

To prove that $\triangle$ ABC and $A'B'C'$ are mutually equiangular.

Proof. On CA lay off $CD = C'A'$, and on CB lay off $CE = C'B'$. Join D and E.

Then	$CA/CD = CB/CE$.	Given
Therefore,	$DE \parallel AB$;	§ 147
whence,	$\angle CDE = \angle A$ and $\angle DEC = \angle B$.	Why?

It follows that the $\triangle$ CDE and CAB are mutually equiangular; Why?

whence also	$\triangle CDE \sim \triangle CAB$;	§ 155
therefore	$AB/DE = CA/CD = CA/C'A'$.	§ 154
But	$CA/C'A' = AB/A'B'$;	Given
whence	$AB/DE = AB/A'B'$,	Why?
and hence	$DE = A'B'$.	Th. A, Cor. 1, § 144
Therefore	$\triangle CDE \cong \triangle C'A'B'$.	Why?

All three of the $\triangle$ CAB, CDE, and $C'A'B'$ are therefore mutually equiangular; hence, in particular, the $\triangle$ ABC and $A'B'C'$ are mutually equiangular.

EXERCISES

1. Show that all equilateral triangles are similar.

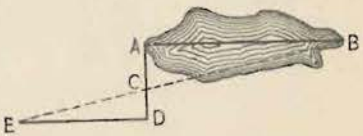
2. Show that all isosceles triangles that have the same vertex angle are similar.

3. Prove that if each of two triangles are similar to a third, they are similar to each other.

4. How high is a house whose shadow is 144 ft. long when that of a gate post 5 ft. high is 12 ft. long? *Ans.* 60 ft.

5. If a triangle be stretched out so that each side becomes one half longer than at the beginning, will the size of the angles be changed? Why?

6. In order to determine the length AB of a lake, set two stakes at convenient points, D and E, such that AD is perpendicular to AB, while DE is perpendicular to AD. Now, from stake E sight the point B and take note of the point C where the line of sight crosses AD. Show how to find AB by measuring ED, DC, and CA.



7. Show that if each of the sides of a triangle is reduced or enlarged in the same scale (§ 21), the angles are unchanged. Thus it is said that *enlargement does not distort a figure.*

8. A triangular tin plate expands in the same ratio along all straight lines when heated. Show that its shape is not changed.

9. Prove that two triangles that have their sides parallel or perpendicular each to each are similar.

10. Prove that the diagonals of any trapezoid divide each other proportionally.

11. Let P and Q be any two points on the sides of an angle AOB. Show that the triangles formed by perpendiculars from P and Q to the sides opposite them, respectively, are similar.

159. Theorem VII. *Two triangles are similar if an angle of the one equals an angle of the other and the including sides are proportional.*

Given the $\triangle\!\!\triangle$ ABC, $A'B'C'$, such that $\angle C = \angle C'$ and $CA/C'A' = CB/C'B'$.

To prove that $\triangle ABC \sim \triangle A'B'C'$.

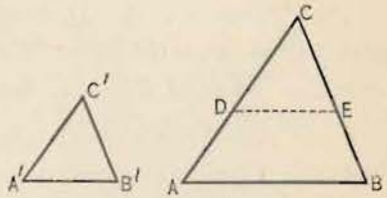


FIG. 117

Proof On CA lay off $CD = C'A'$, and on CB lay off $CE = C'B'$ Join D and E.

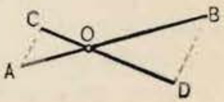
Then	$\triangle CDE \cong \triangle C'A'B'$.	§ 35
Also we have	$CA/CD = CB/CE$;	Given
and hence	$DE \parallel AB$.	§ 147
Therefore	$\angle CDE = \angle A$, and $\angle DEC = \angle B$;	Why?
whence	$\triangle\!\!\triangle$ ABC and DEC are mutually equiangular;	Why?
and	$\triangle\!\!\triangle$ ABC and $A'B'C'$ are mutually equiangular.	Why?
Therefore	$\triangle ABC \sim A'B'C'$.	§ 155

EXERCISES

1. Compare the various conditions under which two triangles are *similar* with those under which two triangles are *congruent.*

2. If the angle at the vertex of a triangle is held fixed while the sides which include the same angle are stretched out by half their length, how do the angles in the resulting triangle compare with those in the original triangle? Why?

3. Two rods AB and CD are hinged together at a point O, the hinge being placed on each rod one third the length of the rod from one end. Show that $AC = BD/2$, whatever the $\angle DOB$ may be. Show also that the ratio AD/CB does not change when $\angle DOB$ changes, if $AB = CD$.



160. Definition. If in the proportion $a/b = c/d$ we have $c = b$, then b (or c) is said to be the **mean proportional** between a and d. Thus, 2 is the mean proportional between 1 and 4, since we have $1/2 = 2/4$.

161. Theorem VIII. ***If, in any right triangle, a perpendicular is drawn from the vertex of the right angle to the hypotenuse, the two right triangles thus formed are similar to each other and to the given triangle.***

Given the right-angled $\triangle ABC$ having its right angle at C. Given also the perpendicular CD drawn from C to the hypotenuse AB.

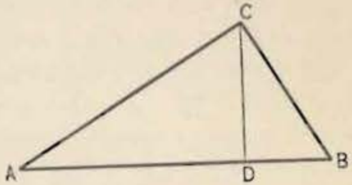


FIG. 118.

To prove that $\triangle$s ADC, CDB, and ABC are similar.

Proof. These $\triangle$s are mutually equiangular; Why?
therefore they are similar. Why?

162. Corollary 1. *In any right triangle the perpendicular from the vertex of the right angle to the hypotenuse is the mean proportional between the segments of the hypotenuse.*

Proof. The $\triangle$s ADC, CDB being similar, their corresponding sides are proportional. Upon comparing the sides AD and CD, in the one with those which correspond to them in the other, namely, the sides CD and DB, we obtain $AD/CD = CD/DB$, which was to be proved.

163. Corollary 2. *If, in any right triangle, a perpendicular is drawn from the vertex of the right angle to the hypotenuse, each side of the right triangle is the mean proportional between the hypotenuse and the segment adjacent to that side.*

Proof. The $\triangle$s ADC and CDB are each similar to $\triangle ABC$.
Hence $AB/AC = AC/AD$ and $AB/CB = CB/DB$.

164. Problem 3. *To find the mean proportional between two lines.*

Given the two lines a and b.

To construct their mean proportional; that is, to construct a line x such that $a/x = x/b$.

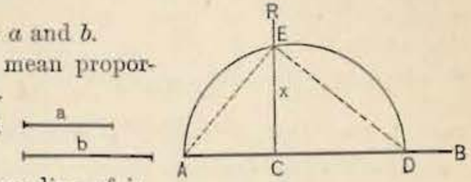


FIG. 119

Construction. Draw a line of indefinite extent AB, and from A lay off $AC = a$. From the point C lay off $CD = b$ (Fig. 119).

On AD as a diameter draw a semicircle.

At C erect $CR \perp AB$, meeting the semicircle at E.

Then CE is the required line x.

Proof. Draw AE and DE. Then $\angle AED$ is a right angle. Why? Complete the proof by means of § 162.

EXERCISES

1. Find the mean proportional between 9 and 4. Answer the same question for 1 and 3. What essential difference exists between the kinds of answers in the two cases? (See § 128.)

2. The altitude drawn to the hypotenuse of a certain right-angled triangle divides the hypotenuse into segments which are of length 16 in. and 4 in., respectively. How long is the altitude? Answer the same question when the segments are each 10 in. long; also when they are respectively 15 in. and 5 in.

3. If a perpendicular is drawn from any point on a circle to a diameter, what relation exists between the perpendicular and the segments into which it divides the diameter? (See § 133.)

4. If one of the sides of a right-angled triangle is three times the other, in what ratio does the perpendicular divide the hypotenuse? *Ans.* 1 : 9

165. Definitions. Similar Polygons. Two *polygons* are said to be **similar** when each may be decomposed into the same number of triangles similar each to each and similarly placed. Thus, in the figure, the polygons $ABCDE$ and $A'B'C'D'E'$ are similar. For, if we draw the lines DA, DB, $D'A'$, $D'B'$, $ABCDE$ is decomposed into three triangles that are similar to the three triangles into which $A'B'C'D'E'$ is decomposed.

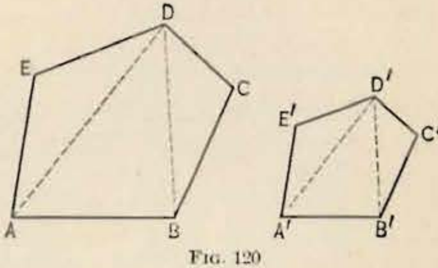


FIG. 120

A **regular** polygon has been defined in § 96 as one which is both equilateral and equiangular.

166. Theorem IX. *Regular polygons of the same number of sides are similar.*

Given the two polygons $ABCDE$ and $A'B'C'D'E'$.

To prove that $ABCDE \sim A'B'C'D'E'$.

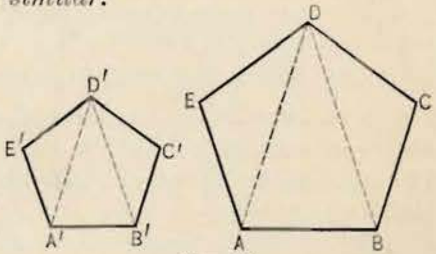


FIG. 121

Proof. Through D draw DA, DB, and through D' draw $D'A'$, $D'B'$, thus dividing each polygon into three triangles.

Then the $\triangle DEA$ and $D'E'A'$ are both isosceles, Why?
and they have the equal vertex angles at E and E'. Given

Therefore $\triangle DEA \sim \triangle D'E'A'$. (Compare Ex. 2, p. 142.)

Now prove that $\triangle DAB \sim \triangle D'A'B'$ by showing that $\angle DAB = \angle D'A'B'$ and that $AD/A'D' = AB/A'B'$. To do this, first show that $\angle EAD = \angle E'A'D'$.

167. Theorem X. *The perimeters of two similar polygons are to each other in the same ratio as any two corresponding sides.*

Given the similar polygons $ABCDE, A'B'C'D'E'$. See Fig. 120

To prove that

$$\frac{\text{perimeter of } ABCDE}{\text{perimeter of } A'B'C'D'E'} = \frac{AB}{A'B'}.$$

Proof. We have

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'}. \qquad \text{Given}$$

Therefore

$$\frac{AB + BC + CD + DE}{A'B' + B'C' + C'D' + D'E'} = \frac{AB}{A'B'}; \qquad \text{Th. II, § 144}$$

that is,

$$\frac{\text{perimeter of } ABCDE}{\text{perimeter of } A'B'C'D'E'} = \frac{AB}{A'B'}.$$

EXERCISES

1. Give the proof of Theorem IX for a regular hexagon; for a regular octagon.

2. In two similar polygons two corresponding sides are 15 and 20 in., respectively. If the perimeter of the first is 5 feet, what is the perimeter of the second? *Ans.* $6\frac{2}{3}$ ft.

3. The perimeters of two similar polygons are to each other as 5 : 8. A side of the first is 1 ft. What is the length of the corresponding side of the other?

4. Prove that the perimeters of similar polygons are in the same ratio as any two corresponding diagonals of the polygons.

5. The perimeter of an equilateral triangle is 3 ft. Find the side of an equilateral triangle whose altitude is one half the altitude of the first triangle. Generalize your result into a theorem relating to similar triangles.

6. Prove that the perimeters of similar triangles are to each other in the same ratio as any two corresponding medians.

PART IV. PROPORTIONAL PROPERTIES OF CHORDS, SECANTS, AND TANGENTS

168. Theorem XI. *If two chords intersect within a circle, the product of the segments of the one is equal to the product of the segments of the other.*

Given the chords AC and BD intersecting at K.

To prove that $KA \cdot KC = KB \cdot KD$.

Proof. Draw AD and BC. In the $\triangle$s AKD and BKC we have $\angle D = \angle C$, each being measured by $\widehat{AB}/2$. § 132

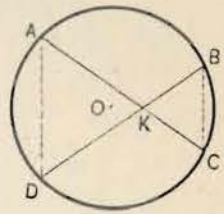


Fig. 122

Likewise $\angle A = \angle B$. Why?

Therefore $\triangle AKD \sim \triangle BCK$; Why?

hence $\dfrac{KA}{KB} = \dfrac{KD}{KC}$; Why?

whence $KA \cdot KC = KB \cdot KD$.

EXERCISES

1. A chord of a circle is divided into two segments of 5 in. and 3 in., respectively, by another chord one of whose segments is 4 in. How long is the second chord? *Ans.* $7\frac{3}{4}$ in.

2. The greatest distance to a chord 8 in. long from a point on its intercepted (minor) arc is 2 in. What is the diameter of the circle?

3. P is a fixed point within a circle through which (point) a chord passes. As the chord swings round P as a pivot, what can be said of the segments of all new chords thus obtained?

4. If two circles intersect and through any point in their common chord two other chords be drawn, one to each circle, prove that the product of the segments of one chord is equal to the product of the segments of the other.

169. Theorem XII. *If from a point without a circle a secant and a tangent are drawn, the tangent is the mean proportional between the entire secant and its exterior segment.*

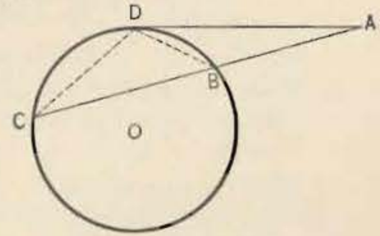


Fig. 123

To prove that $AC/AD = AD/AB$.

Outline of proof. Draw BD and DC.

Then $\angle A = \angle A$, Iden.

and $\angle C = \angle ADB$, since each is measured by $\frac{1}{2}\widehat{DB}$. Why?

Therefore $\triangle ACD \sim \triangle ABD$, § 155

and hence $AC/AD = AD/AB$. Why?

Note. If in Fig. 123, AC is allowed to swing around A as a pivot, the tangent AD meanwhile remaining fixed, we have throughout the motion, $AC \cdot AB = \overline{AD}^2$. This, in fact, is what Theorem XII says. But, since AD remains fixed, this means that the product $AC \cdot AB$ remians *constant* throughout the motion. This fact may be stated as follows:

170. Theorem XIII. *If from a fixed point without a circle any two secants are drawn, the product of one secant and its external segment is equal to the product of the other secant and its external segment.*

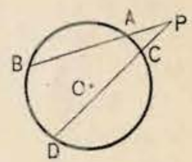


Fig. 124

EXERCISES

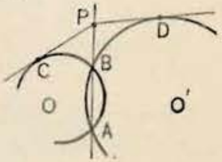
1. A secant is drawn from a point without a circle in such a way that its whole length is 9 in., while the part cut off within the circle is 4 in. What is the length of the tangent to the circle from the same point? *Ans.* $3\sqrt{5}$ in.

2. Two secants are drawn from the same point without a circle. One is 11 in. long and its external segment is 4 in. The external segment of the second is 2 in. What is the length of the second secant?

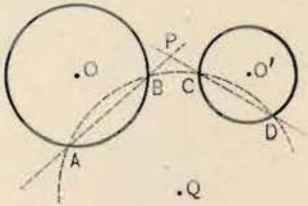
3. One of two secants meeting without a circle is 18 in. long, while its external segment is 4 in. The other secant is divided into equal parts by the circumference. Find the length of the second secant.

4. A point P is 12 in. from the center of a circle whose radius is 7 in. A secant is drawn from P. Find the product of the entire secant and its external segment.

5. Prove that tangents drawn to two intersecting circles from a point on their common chord are equal.



6. Given two non-intersecting circles O and O', draw any circle $ABCD$ intersecting both of them. Draw the two common chords AB and CD, and extend them to meet at P. Show that tangents drawn from P to the two circles O and O' are equal.



7. Show that the diameter of a circular porch column can be found by measuring certain lines entirely outside the column. Can this still be done if over half the face of the column is buried in cement?

171. Definition. A line segment is said to be divided in **extreme and mean ratio** when one of its segments is a mean proportional between the whole line and the other segment. In Fig. 125, C is so located that $AB/AC = AC/CB$, so that AB is divided by C in extreme and mean ratio.

FIG. 125

NOTE. The division of a line in extreme and mean ratio has been called the *golden section* or division in *golden mean*. It was observed by the ancients that artistic effects frequently result from its use. Thus, a rectangular picture frame will usually give the best effect if its length and width are in the ratio just described; that is, in the ratio of AC to CB in Fig. 125. A similar remark applies to the height of the back of a chair as compared with the length of its legs.

172. Problem 4. *To divide a given line segment in extreme and mean ratio.*

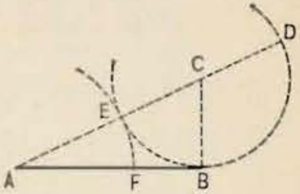


FIG. 126

Given the line AB.

Required to divide AB in extreme and mean ratio; that is, to determine a point F such that $AB/AF = AF/FB$.

Construction. At B draw $BC \perp AB$ and equal to half of AB.

With C as center and CB as radius draw a circle.

Draw AC meeting the circle at E and D.

On AB take $AF = AE$.

Then F is the point of division required.

Proof. $AD/AB = AB/AE$; Why?

hence $\dfrac{AD-AB}{AB} = \dfrac{AB-AE}{AE}$. Why?

But $AB = ED$, and $AF = AE$; Why?

therefore $AF/AB = FB/AF$, or $AB/AF = AF/FB$. Why?

PART V. SIMILAR RIGHT TRIANGLES. TRIGONOMETRIC RATIOS

173. Similar Right Triangles. The importance of a special study of similar right triangles results from the simplicity and usefulness of the result, § 157, that **two right triangles are similar if an acute angle of one is equal to an acute angle of the other.** This proposition may now be restated as follows:

174. Theorem XIV. ***If two right triangles have an acute angle of one equal to an acute angle of the other, their corresponding sides are in the same ratios.***

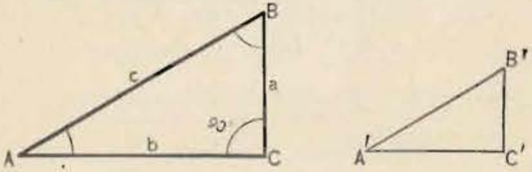


Fig. 127

Given the rt. $\triangle\triangle$ ABC and $A'B'C'$, with $\angle A = \angle A'$.

To prove that their corresponding sides are in the same ratios.

Proof. $\triangle ABC \sim \triangle A'B'C'$; § 157

hence the corresponding sides are in the same ratios;

that is: $\dfrac{BC}{AB} = \dfrac{B'C'}{A'B'}$; $\dfrac{AC}{AB} = \dfrac{A'C'}{A'B'}$; $\dfrac{BC}{AC} = \dfrac{B'C'}{A'C'}$. *D*, § 144

175. Corollary 1. *If an acute angle of a right triangle is known, the ratios of the sides are all determined.*

176. Corollary 2. *If the ratio of any pair of sides of a right triangle is given, the acute angles are determined.*

NOTE. The study of the use of these ratios is called **Trigonometry.** By their use, any part of a right triangle can be found, if any two parts, not both angles, are given, besides the right angle. This process is called **solving the triangle.**

177. Definitions. Let $\angle A$, Fig. 128, be a known acute angle of a right triangle, in which $\angle C$ is a right angle; and let us denote the sides opposite $\angle A$, $\angle B$, $\angle C$, by a, b, c, respectively.

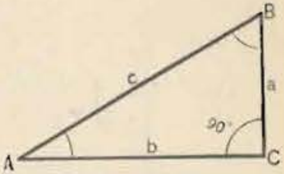


Fig. 128

Then the ratios a/b, a/c, b/c are all determined, by § 175. These ratios are named as follows:

(1) a/c is called the **sine** of the angle A;

(2) b/c is called the **cosine** of the angle A;

(3) a/b is called the **tangent** of the angle A.

That is, for an acute angle of a **right triangle**:

(1) the **sine** of the angle = side opposite it ÷ hypotenuse;

(2) the **cosine** of the angle = side adjacent it ÷ hypotenuse;

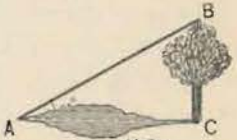
(3) the **tangent** of the angle = side opposite ÷ side adjacent.

These ratios are called the **trigonometric ratios.**

178. Values of the Ratios. Table. The values of the three ratios mentioned in § 177 can be obtained, approximately, from a good figure, for any acute angle A.

These values are tabulated on p. 155, to three places of decimals, for every angle from 0° to 90°, at intervals of 1°. The student may check the accuracy of any entry in this table by drawing an accurate figure with a protractor, actually measuring the sides of it, and then calculating their ratios.

EXAMPLE. The length of the shadow of a tree CB is 44 ft. when the angle CAB between the shadow CA and the line AB is 31°. Find the height of the tree.

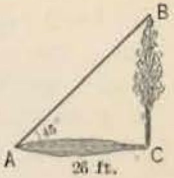


[SOLUTION. Since BC/AC is the tangent of 31°, by § 177, we look in the table, in the column headed tangent, and opposite 31°. This gives the value .601; hence $BC/AC = .601$; but $AC = 44$ ft.; it follows that

$$BC = AC \times .601 = 44 \times .601 = 26.444 \text{ (ft.)}.]$$

EXERCISES

1. How large, in degrees, is an acute angle whose tangent is 1?



2. The shadow of a tree is 26 ft. long when the angle of elevation of the sun ($\angle CAB$ in the figure) is 45°. How tall is the tree?

3. One side of a right triangle is 2 in.; the adjacent angle is 42°; determine the remaining side and the hypotenuse, and check by measurement from an accurate figure. *Ans.* side = 1.8 in.; hyp. = 2.69 in.

4. One side of a right triangle is 2 in. and the opposite angle is 42°; determine the remaining side and hypotenuse. Check by measurement in a figure.

5. The hypotenuse of a right triangle is 28 in.; one angle is 32°. Determine the two perpendicular sides. Check.

6. To determine the height of a tree OA standing in a level field the distance $OB = 100$ ft. from the base O of the tree to a point B in the field is measured, and the angle OBA is then found to be 37°. Find approximately the height by measurement in a reduced figure, and by the table, p. 155.

7. Measure two adjacent edges of your study table. Find the angles that the diagonal makes with the edges, (*a*) by drawing an accurate figure and measuring the angle with a protractor; (*b*) by use of the table (p. 155).

8. The tread of a step on a certain stairway is 10 in. wide; the step rises 7 in. above the next lower step. Find the angle at which the stairway rises, (*a*) by a protractor from an accurate figure; (*b*) from the table, p. 155. *Ans.* 35° (nearly).

9. What angle does a rafter make with the plate beam (Fig., Ex. 2, p. 43), if the roof is "half-pitch"; if the pitch of the roof is 1/3. *Ans.* 45°; nearly 34°.

[The *pitch* of a roof is the rise divided by the *entire* span.]

[The abbreviation *hyp* means *hypotenuse*; *adj* means the *side adjacent* to the angle; *opp* means the *side opposite* the angle. See also the larger table, p. iii.]

Angle	Sine (opp/hyp)	Cosine (adj/hyp)	Tangent (opp/adj)	Angle	Sine (opp/hyp)	Cosine (adj/hyp)	Tangent (opp/adj)
0°	.000	1.000	.000	**45°**	.707	.707	1.000
1°	.017	1.000	.017	46°	.719	.695	1.036
2°	.035	.999	.035	47°	.731	.682	1.072
3°	.052	.999	.052	48°	.743	.669	1.111
4°	.070	.998	.070	49°	.755	.656	1.150
5°	.087	.996	.087	**50°**	.766	.643	1.192
6°	.105	.995	.105	51°	.777	.629	1.235
7°	.122	.993	.123	52°	.788	.616	1.280
8°	.139	.990	.141	53°	.799	.602	1.327
9°	.156	.988	.158	54°	.809	.588	1.376
10°	.174	.985	.176	**55°**	.819	.574	1.428
11°	.191	.982	.194	56°	.829	.559	1.483
12°	.208	.978	.213	57°	.839	.545	1.540
13°	.225	.974	.231	58°	.848	.530	1.600
14°	.242	.970	.249	59°	.857	.515	1.664
15°	.259	.966	.268	**60°**	.866	.500	1.732
16°	.276	.961	.287	61°	.875	.485	1.804
17°	.292	.956	.306	62°	.883	.469	1.881
18°	.309	.951	.325	63°	.891	.454	1.963
19°	.326	.946	.344	64°	.899	.438	2.050
20°	.342	.940	.364	**65°**	.906	.423	2.145
21°	.358	.934	.384	66°	.914	.407	2.246
22°	.375	.927	.404	67°	.921	.391	2.356
23°	.391	.921	.424	68°	.927	.375	2.475
24°	.407	.914	.445	69°	.934	.358	2.605
25°	.423	.906	.466	**70°**	.940	.342	2.747
26°	.438	.899	.488	71°	.946	.326	2.904
27°	.454	.891	.510	72°	.951	.309	3.078
28°	.469	.883	.532	73°	.956	.292	3.271
29°	.485	.875	.554	74°	.961	.276	3.487
30°	.500	.866	.577	**75°**	.966	.259	3.732
31°	.515	.857	.601	76°	.970	.242	4.011
32°	.530	.848	.625	77°	.974	.225	4.331
33°	.545	.839	.649	78°	.978	.208	4.705
34°	.559	.829	.675	79°	.982	.191	5.145
35°	.574	.819	.700	**80°**	.985	.174	5.671
36°	.588	.809	.727	81°	.988	.156	6.314
37°	.602	.799	.754	82°	.990	.139	7.115
38°	.616	.788	.781	83°	.993	.122	8.144
39°	.629	.777	.810	84°	.995	.105	9.514
40°	.643	.766	.839	**85°**	.996	.087	11.430
41°	.656	.755	.869	86°	.998	.070	14.301
42°	.669	.743	.900	87°	.999	.052	19.081
43°	.682	.731	.933	88°	.999	.035	28.636
44°	.695	.719	.966	89°	1.000	.017	57.290
45°	.707	.707	1.000	**90°**	1.000	.000	——

179. Theorem XV. *Corresponding altitudes divide any two similar triangles into two corresponding pairs of similar right triangles.*

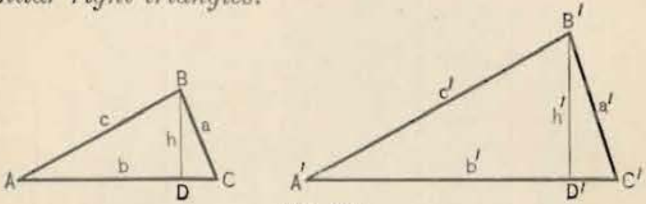


FIG. 129

Given the two similar triangles ABC and $A'B'C'$; and given the corresponding altitudes BD and $B'D'$.

To prove that $\triangle ABD \sim \triangle A'B'D'$, and $\triangle DCB \sim \triangle D'C'B'$.

Outline of proof. Show that $\triangle ABD \sim \triangle A'B'D'$ by showing that $\angle A = \angle A'$. (§§ 158, 174.)

180. Corollary 1. *Any two similar polygons may be subdivided into corresponding pairs of similar right triangles.*

NOTE. Division of a triangle into right triangles is often useful.

EXAMPLE. In Fig. 129, suppose $c = 10$ in., $\angle A = 30°$, $\angle B = 85°$. Find $\angle C$, and sides a and b.

SOLUTION: First find $\angle C = 65°$. Then $h = c \times$ cosine of $30° = 10 \times \frac{1}{2} = 5$. Hence $a = h \div$ sine of $C = 5 \div .906 = 5.52$. Again $AD = c \times$ cosine of $30° = 8.66$; and $DC = a \times$ cosine of $65° = 2.33$; hence $b = AD + DC = 10.99$.

EXERCISES

1. The base of a certain isosceles triangle is 10 in., and the angle at the vertex is 40°. Find the size of one of the equal angles; find the length of one of the equal sides (a) by measurement, (b) using the table, p. 155.

2. The diagonal of a certain rectangle is 4 ft. One side is 2 ft. Find the angle the diagonal makes with that side, (a) by measurement, (b) using the table, p. 155.

MISCELLANEOUS EXERCISES. CHAPTER III

[The Exercises that involve the use of the trigonometric ratios are starred, *.]

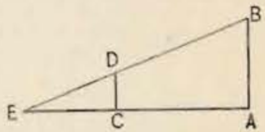
1. A building casts a shadow 64 ft. long. A lamp-post 8 ft. high at one corner of the building casts at the same time a shadow 9 ft. long. How high is the building?

2. Show how to find three fifths of a given line; five sevenths.

3. A circle is inscribed in an isosceles triangle. Prove that the triangle formed by joining the points of contact is also isosceles.

4. If each of two polygons is similar to a third polygon, they are similar to each other. Prove this statement.

5. Prove that in an inscribed quadrilateral the product of the segments of one diagonal is equal to the product of the segments of the other.



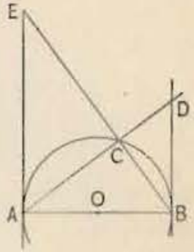
6. In the adjoining figure, AB represents a vertical pole, and CD a vertical stake, so that D and B are in the same line of sight from a point E on the level ground ECA. What measurements must be taken to find AB?

7. If the non-parallel sides of a trapezoid are extended to meet in a point P, the lengths of the extensions are proportional to the lengths of the original sides. State a proof.

8. If in any triangle, a line is drawn parallel to the base, any line through the vertex divides the base and the parallel into segments that are in the same ratio. State a proof.

9. Show that any two altitudes h and h' of a triangle are inversely proportional to the sides a and a' to which they are perpendicular; that is, $h/h' = a'/a$.

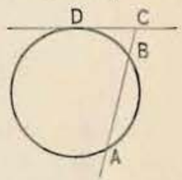
10. Let AE and BD be the tangents at the ends of a diameter AB of a circle O. Draw any line through A, and suppose it cuts the circle at C and meets BD at D. Let E be the point of intersection of AE and BC produced. Show that three similar right triangles are formed; and show that AB is a mean proportional between AE and BD.



11. Construct a fourth proportional to three lines, 3 in., 5 in., and 1 in. long, respectively. Show that the resulting line must be $\frac{5}{3}$ in. long.

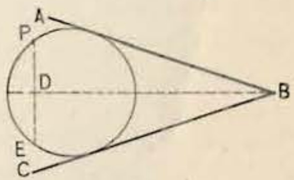
12. In general, construct a fourth proportional to two given line segments a and b and the unit line segment. Show that the resulting line has a length b/a in terms of the unit. [*Geometric construction for division.*]

13. Show how to construct a circle through two given points and tangent to a given line.



[HINT. Draw a line through the given points A and B, meeting the given line at C. Then find a mean proportional CD between AC and BC. *Two* circles are possible according as CD is laid off from C in one direction or the other along the given line.]

14. Show how to constuct a circle tangent to two given lines and passing through a given point P.



[HINT. Draw BD bisecting $\angle ABC$. Draw through the given point P a line $PD \perp BD$ and extend it to E, making $DE = PD$. Now apply Ex. 13 to draw a circle through P and E and tangent to AB. How many solutions are possible?]

15. Show how to inscribe in a given circle a triangle similar to a given triangle.

16. Show how to circumscribe about a given circle a triangle similar to a given triangle.

17. Construct a fourth proportional to three lines, 1 in., 2.5 in., and 3.5 in. long. Show that the resulting line must be 2.5×3.5 in. long.

18. Give a *geometric construction for multiplying any two numbers.*

[HINT. If $x = ab$, then $1/a = b/x$.]

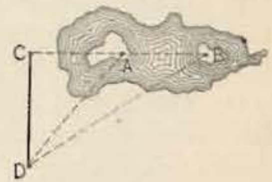
19. Give a *geometric construction for enlarging any line segment* in the ratio $5:7$. Use this construction to draw a triangle similar to a given triangle, with its sides enlarged $5:7$.

20. Show how to enlarge (or reduce) any line segment in the ratio of two given numbers (or two given line segments).

21. Show how to construct, on a given line segment as one side, a polygon similar to a given polygon.

22. Show that if two circles are tangent externally, any line through their point of tangency forms chords of the two circles that are proportional to their radii. State and prove the analogous theorem when the circles are tangent internally.

23. In order to find the distance between two islands A and B in a lake, what distances and angles must be measured in the adjoining figure? Compare Ex. 6, p. 142. Show that the figure can be extended, so as to find AB by measuring *distances* only.



24. If a line is drawn parallel to the parallel bases of a trapezoid, show that the segment cut off on it between one side and one diagonal is equal to that cut off by the other side and the other diagonal.

25. The bases of a trapezoid are 10 in. and 15 in. long, respectively, and the altitude is 8 in. Find the altitude of the triangle formed on the smaller base by extending the nonparallel sides until they meet. *Ans.* 16 in.

26. Construct a mean proportional to two lines 3 in. and 4 in. long, respectively. Construct a mean proportional to two lines, each of which is 2 in. long.

27. Construct a mean proportional to two lines 1 in. and 5 in. long, respectively. How long is it?

28. Show how to construct the square root of any given number n by finding the mean proportional between a line of unit length and a line n units long. [*Geometric construction for square root.*]

29. The total length of a certain secant drawn from a point P to a circle is 10 in.; its external segment is 4 in. Find the length of the tangent drawn from P.

30. Taking the radius R of the earth as 4000 mi., find how far from a lighthouse 150 ft. high the light is visible. A closer value of R is 3963 mi.; is the answer changed seriously by using this more accurate value of R?

* 31. If one side of a right triangle is double the other, in what ratio is the hypotenuse divided by the altitude drawn to it? What are the angles of the triangle?

32. If one chord of a circle is bisected by another, show that either segment of the first is a mean proportional between the segments of the other.

33. The greatest distance from a chord 10 in. long to its intercepted arc is 3 in. Find the radius of the circle.

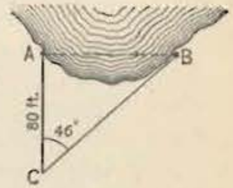
34. A curved pane of glass to fit a window in a round tower is bent in the arc of a circle. If the width of the frame is 30 in. and if the greatest distance from the glass to a horizontal line joining its edges is 1.5 in., find the radius of the arc.

* 35. The radius of a circle is 7 ft. What angle will a chord of the circle 11 ft. long subtend at the center? Check by measurement in a reduced figure.

36. A railroad curve is to have a radius of 250 ft. What is the greatest distance from the track to a chord 100 ft. long?

37. A railroad curve turns in the arc of a circle; the greatest distance from the track to a chord 100 ft. long is 7 ft.; find the radius of the arc. *Ans.* 182^+ ft.

* 38. The width of the gable of a house is 34 ft. The height of the house above the eaves is 15 ft.; find the length of the rafters and the angle of inclination of the roof. Find the pitch of the roof. See Ex. 2, p. 43, and Ex. 9, p. 154.



* 39. To find the distance across a lake between two points A and B, a surveyor measured off 80 ft. on a line AC perpendicular to AB; he then found $\angle ACB = 46°$. Find AB.

* 40. A kite string is 250 ft. long and makes an angle of 40° with the level ground. Find (approximately) the height of the kite above the ground, neglecting the sag in the string.

* 41. The shadow of a vertical 10-foot pole is 14 ft. long. What is the angle of elevation of the sun? *Ans.* About 35.5°.

* 42. A chord of a circle is 21.5 ft. long, and the angle which it subtends at the center is 41°. Find the radius of the circle.

* 43. The base of an isosceles triangle is 324 ft., the angle at the vertex is 64° 40′. Find the equal sides and the altitude.

* 44. The base of an isosceles triangle is 245.5 and each of the base angles is 68° 22′. Find the equal sides and the altitude.

* 45. The altitude of an isosceles triangle is 32.2 and each of the base angles is 32° 42′; find the sides of the triangle.

* 46. Find the length of a side of an equilateral triangle circumscribed about a circle of radius 15 in.

* 47. Show that the sine of the angle A in Fig. 129 is $h \div c$, and that the sine of C is $h \div a$. Hence show that the *sines of the angles A and C are proportional to the sides a and c opposite them.* [SINE LAW.]

CHAPTER IV

AREAS OF POLYGONS. PYTHAGOREAN THEOREM

181. Area of a Rectangle. The fundamental principle, mentioned in the Introduction (§ 25), that **the area of a rectangle is equal to the product of its base by its height,** will be presupposed in what follows in the present chapter.

The principle states that *if the base and the height of a rectangle are, respectively,* a *and* b, *then its area (in terms of the corresponding square unit) is* $a \cdot b$.

The following corollaries result from this principle:

182. Corollary 1. *The area of a square is equal to the square of its side.*

183. Corollary 2. *The areas of two rectangles are to each other as the products of their bases and altitudes.*

184. Corollary 3. *Two rectangles that have equal altitudes are to each other as their bases; two rectangles that have equal bases are to each other as their altitudes.*

185. Definition. Whenever two geometric figures have the same area they are said to be **equal in area,** or *equivalent.* The equality in area of two figures is denoted by the symbol =. Thus, the equation $\triangle ABC = \triangle A'B'C'$ means that the two triangles have equal areas.

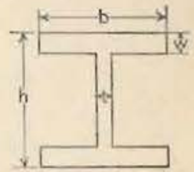
EXERCISES

1. How many tiles each 8 in. square will it take to tile a floor 30 ft. long and 18 ft. wide?

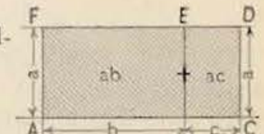
2. On a sheet of squared paper ruled in tenths of an inch (see p. 23), how many small squares are there in one square inch?

3. Compare the areas of two rectangles whose altitudes are equal but whose bases are respectively 10 in. and 7 in.

4. The area of a rectangle is 400 sq. ft. and its altitude is 20 ft. What is the altitude of another rectangle having the same base but whose area is 300 sq. ft.?



5. The accompanying figure represents the cross section of a steel beam. The dimensions in millimeters are: $b=96$, $w=12$, $h=192$, $t=8$. Find the area of the cross section.

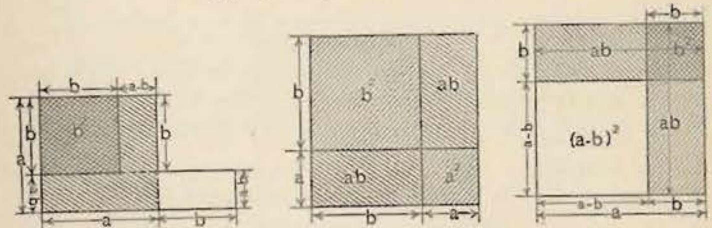


6. Show that the adjoining figure illustrates the algebraic identity

$$a(b+c) = ab + ac.$$

7. Show that the following figures illustrate the algebraic identities

(1) $(a+b)(a-b) = a^2 - b^2$; (2) $(a+b)^2 = a^2 + 2\,ab + b^2$; (3) $(a-b)^2 = a^2 - 2\,ab + b^2$.



(1) (2) (3)

8. Draw a figure to illustrate each of the identities

(1) $a(b-c) = ab - ac$; (2) $(x+a)(x+b) = x^2 + (a+b)x + ab$.

9. Draw a rectangle 5 in. long by 6 in. wide. Show that either diagonal divides the rectangle into two congruent right triangles. Find the area of each of these triangles.

Ans. 15 sq. in.

10. Find the area of a right triangle whose two sides are 4 in. and 7 in. long, respectively.

186. Theorem I. *The area of a parallelogram is equal to the product of its base by its altitude.*

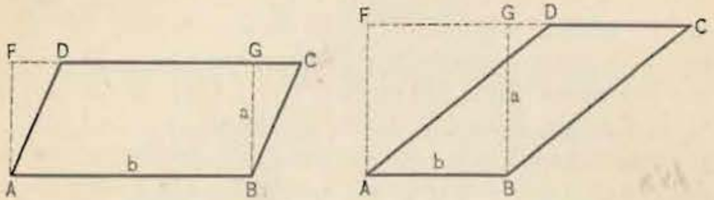


FIG. 130

Given the parallelogram $ABCD$ with b its base and a its altitude.

To prove that the area of the parallelogram $ABCD$ is equal to $a \cdot b$.

Proof. Draw $AF \perp CD$ prolonged. Then $ABGF$ is a rectangle having the base b and the altitude a.

In the right $\triangle$ AFD and BGC we have $AF = BG$ and $AD = BC$. Why?

Therefore $\triangle AFD \cong \triangle BGC$. Why?

Taking away $\triangle AFD$ from the figure $ABCF$, the parallelogram $ABCD$ remains.

Taking away $\triangle BGC$ from the same figure, the rectangle $ABGF$ remains.

Therefore $\square ABCD =$ rectangle $ABGF$.

But rectangle $ABGF = ab$, § 181

whence, $\square ABCD = ab$.

187. Corollary 1. (*a*) *Two parallelograms are to each other as the products of their bases and altitudes.*

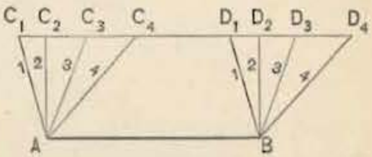
(*b*) *Two parallellograms that have equal bases and equal altitudes are equal in area.*

188. Corollary 2. *Two parallelograms that have equal altitudes are to each other as their bases; two parallelograms that have equal bases are to each other as their altitudes.*

EXERCISES

1. In a certain parallelogram the acute angle included between the sides is 30°. If the base and altitude are respectively 14 in. and 10 in., what is the area? Answer the same question in case the included angle is 60°.

2. In the accompanying figure are a number of parallelograms each having the same base and altitude. Compare their areas. Does the area of the parallelogram depend upon the angles included by its sides?



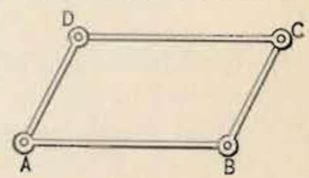
3. Prove that the lines joining the middle points of the opposite sides of a parallelogram divide it into four parallelograms that are equal in area.

4. Construct a parallelogram double a given parallelogram and equiangular to it. How many solutions are possible?

5. Construct a parallelogram double a given parallelogram and having one of its angles equal to a given angle.

6. What is the locus of the intersection of the diagonals of a parallelogram whose base is fixed and whose area is constant?

7. $ABCD$ is a jointed parallelogram frame, that is, it consists of four pieces of stiff material hinged at each of the points A, B, C, and D, and such that *side* AD = *side* BC and *side* AB = *side* DC. If the base AB is held fixed while DC is raised and lowered into various positions, will the areas of the various parallelograms be changing? If so, what will be the greatest area obtainable and what the least, provided that $AB = 6$ in. and $BC = 4$ in.?



189. Theorem II. ***The area of a triangle is equal to one half the product of its base and its altitude.***

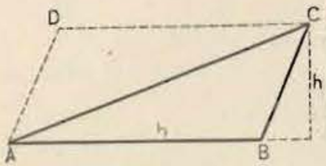


FIG. 131

Given the triangle ABC, having the base b and the altitude h.

To prove that the area of the triangle ABC is equal to $\frac{1}{2}hb$.

Proof. Construct the $\square\ ABCD$.

Then	$\square\ ABCD = hb.$	§ 186
But	$\triangle\ ABC = \frac{1}{2}\square\ ABCD.$	§ 82
Therefore	$\triangle\ ABC = \frac{1}{2}hb.$	

190. Corollary 1. (*a*) *Two triangles are to each other as the products of their bases and altitudes.*

(*b*) *Two triangles that have equal bases are to each other as their altitudes.*

(*c*) *Two triangles that have equal altitudes are to each other as their bases.*

(*d*) *Two triangles that have equal bases and equal altitudes are equal in area.*

EXERCISES

1. What is the area of a right triangle whose sides are respectively 3 in. and 5 in.?

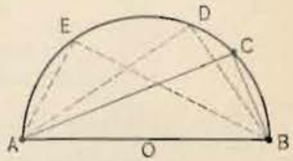
2. Draw several triangles having equal bases and equal altitudes, as an illustration of § 190 (*d*). Are such triangles necessarily congruent?

3. Compare two triangles with equal altitudes if the base of the first is two thirds that of the second.

4. Show that any median of a triangle divides it into two equal triangles. Is the same true of any altitude? Give reason.

5. What is the locus of the vertices of all triangles having a common base and the same area?

6. An ordinary elastic rubber band is stretched out and placed around two pins A and B which are stuck into a board. The pins are placed at the extremities of a diameter of a circle, as indicated in the figure. One of the halves of the band is now thrust aside by means of a pencil point so that the band becomes stretched out into the form of an elastic triangle having one of its vertices on the circle at C. If the pencil be now moved about on the circumference, the band meanwhile slipping over the point, various triangles are formed, such as those indicated by dotted lines in the figure. Do these triangles all have the same area? If not, what is the greatest area obtainable for any triangle, and what the least, provided that the distance between the pins is 6 in.?



7. Show that the diagonals of a parallelogram divide it into four equal triangles.

8. If a line is drawn from the vertex of a triangle to any point P in the base, show that the areas of the two triangles formed are to each other as the segments of the base made by P.

9. Prove that if the middle points of two sides of a triangle are joined, a triangle is formed whose area is one fourth the area of the given triangle.

10. Show that the area of a rhombus is equal to one half the product of its diagonals.

11. Prove that the area of an isosceles right triangle is equal to one fourth of the area of the square erected upon its hypotenuse.

191. Theorem III. *The area of a trapezoid is equal to the product of its altitude and one half the sum of its bases.*

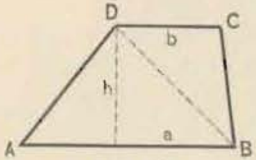


FIG. 132

Outline of Proof. Draw the diagonal BD.

Then $\triangle ABD = ah/2$ and $\triangle BCD = bh/2$; Why?

hence $$ABCD = \frac{ah}{2} + \frac{bh}{2} = h\left(\frac{a+b}{2}\right).$$

192. Corollary 1. *The area of a trapezoid is equal to the product of its altitude and the line joining the mid-points of the non-parallel sides.*

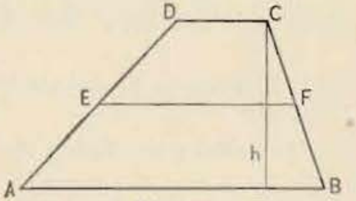


FIG. 133

[HINT. To prove area of $ABCD = h \cdot EF$. It may be easily shown that $EF = (AB + DC)/2$. See § 89.]

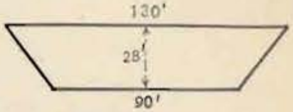
NOTE. The **median** of a trapezoid is a straight line that joins the middle points of the non-parallel sides. Thus, in Fig. 133, EF is the median of the trapezoid $ABCD$.

The corollary of § 192 is frequently stated in the following form: *The area of a trapezoid is equal to the product of its altitude and its median.*

EXERCISES

1. Find the area of the trapezoid whose bases are 6 in. and 4 in. respectively and whose altitude is 3 in. *Ans.* 15 sq. in.

2. Find the area of a trapezoid whose median is 8 in. and whose altitude is 6 in.



3. An excavation for a railway track is 28 ft. deep, 130 ft. wide at the top, and 90 ft. wide at the bottom. What is the area of its cross section?

193. Theorem IV. *Two triangles that have an acute angle of the one equal to an acute angle of the other are to each other as the products of the sides including the equal angles.*

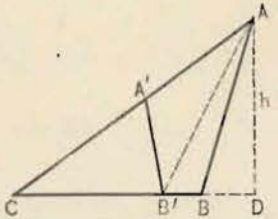


FIG. 134

Given the $\triangle$s ABC and $A'B'C$ having the $\angle C$ common.

To prove that $$\frac{\triangle ABC}{\triangle A'B'C} = \frac{AC \cdot BC}{A'C \cdot B'C}.$$

Proof. Draw AB'. Then, since the triangles ABC, $AB'C$ have the same altitude h,

$$\frac{\triangle ABC}{\triangle AB'C} = \frac{BC}{B'C}, \qquad \text{§ 190, } (c)$$

and likewise $$\frac{\triangle AB'C}{\triangle A'B'C} = \frac{AC}{A'C}.$$

Multiplying, we obtain $$\frac{\triangle ABC}{\triangle A'B'C} = \frac{AC \cdot BC}{A'C \cdot B'C}.$$

194. Theorem V. *Similar triangles are to each other as the squares of any two corresponding sides.*

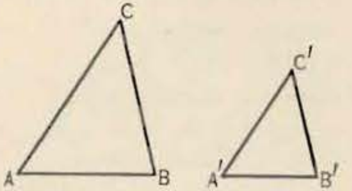


FIG. 135

Given the similar $\triangle ABC$ and $A'B'C'$.

To prove that $$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Proof. $\angle A = \angle A'$. Why ?

Therefore $$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{AC \cdot AB}{A'C' \cdot A'B'}.$$ § 193

But $$\frac{AC}{A'C'} = \frac{AB}{A'B'}.$$ Why ?

Therefore $$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{AB \cdot AB}{A'B' \cdot A'B'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

195. Corollary 1. *The areas of two similar polygons are to each other as the squares of any two corresponding sides.*

[HINT. Divide the polygons up into the same number of similar triangles, as in Fig. 120, § 165, then apply Theorem V, together with § 144, Theorem H.]

NOTE. Since any two corresponding lines in two similar figures are proportional to two corresponding sides, it follows that *the areas of any two similar polygons are to each other as the squares of any two corresponding lines.* Compare, in particular, Exs. 4 and 6, p. 171.

EXERCISES

1. Compare the areas of two similar triangles whose corresponding sides are in the ratio 3 : 4. *Ans.* 9 : 16.

2. Draw on heavy cardboard two triangles whose sides are in the ratio 1 : 2. Cut these triangles out and weigh each of them. Show that their weights should be in the ratio 1 : 4.

3. The areas of two similar triangles are 100 square feet and 64 square feet respectively. Compare the lengths of their corresponding sides. Answer the same question when the given areas are respectively 31 square feet and 17 square feet. What distinction is to be made between the two cases ?

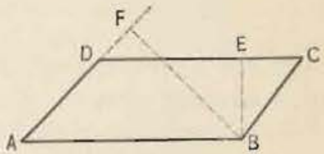
4. Prove that the areas of two similar triangles are to each other as the squares of any two corresponding altitudes.

5. One side of a polygon measures 8 feet and its area is 120 square feet. The corresponding side of a certain similar polygon measures 20 feet. What is the area of the second polygon ?

6. Prove that the areas of any two similar polygons are to each other as the squares of any two corresponding diagonals.

7. If one square is double another, what is the ratio of their sides ?

8. In the figure, BE is perpendicular to CD while BF is perpendicular to AD. Prove that $AB \cdot BE = AD \cdot BF$.



9. In what ratio must the altitude of a triangle be divided by a line drawn parallel to the base in order that the area of the triangle may be divided into two equal parts ? *Ans.* $1 : (\sqrt{2} - 1)$.

[HINT. Call h the altitude of the given triangle and let x be the altitude of the small triangle cut off by the parallel to the base. Form an equation between h and x, solve for x and then form $x / (h - x)$.]

196. Theorem VI. The Pythagorean Theorem. *The square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the two sides.*

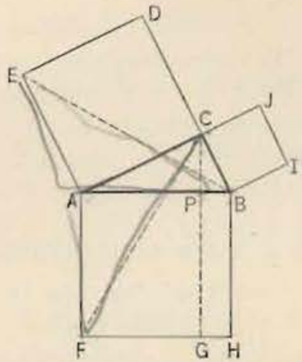


FIG. 136

Given the rt. $\triangle ABC$ having AB as its hypotenuse.

To prove that $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

First Proof. Draw $CP \perp AB$ and prolong it to meet FH at G. Draw CF and BE.

Since $\angle ACD$ and ACB are rt. $\angle$s, DCB is a straight line.

Similarly, ACJ is a straight line.

In $\triangle$s ABE and AFC, $AF = AB$ and $AE = AC$. Why?

Also $\angle BAE = \angle CAF$, since each consists of a right angle plus the angle BAC. Therefore $\triangle ABE \cong \triangle AFC$. Why?

But the rectangle $AFGP = 2 \cdot \triangle AFC$, since both have the same base AF and the same altitude FG (or AP).

Likewise, the square $ACDE = 2 \cdot \triangle ABE$, since each has the same base AE and a common altitude AC.

Therefore rectangle $AFGP =$ square $ACDE$. Ax. 9

Show similarly that rectangle $BPGH =$ square $CBIJ$.

Therefore, $AFGP + BPGH = ACDE + CBIJ$. Ax. 1

That is, $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

Second Proof. By § 163,

$$\overline{AC}^2 = AB \cdot AP; \text{ and } \overline{CB}^2 = AB \cdot PB;$$

hence $$\overline{AC}^2 + \overline{CB}^2 = AB(AP + PB) = \overline{AB}^2.$$

NOTE 1. This second proof is the same in principle as the first, for $\overline{AC}^2 =$ square $ACDE$, and $AB \cdot AP =$ rectangle $AFGP$ and $AB \cdot PB =$ rectangle $PGHB$. Each proof consists essentially in showing that $\overline{CB}^2 = AB \cdot PB$ and $\overline{AC}^2 = AB \cdot AP$.

The second proof might have been given in connection with § 163.

NOTE 2. It should be observed that Theorem VI, though relating specifically to *areas*, furnishes at once a rule for finding the length of the hypotenuse of any right triangle when the lengths of its sides are known. Thus, if a and b are the sides, the hypotenuse h will be determined by the formula $h = \sqrt{a^2 + b^2}$. Similarly, the theorem furnishes a rule for finding either side of a right triangle when the hypotenuse and the other side are known, the formula then being $a = \sqrt{h^2 - b^2}$. These two formulas are of great value in mathematics.

NOTE 3. Aside from its scientific value, Proposition VI is of great interest historically. Though its origin is not known exactly, it is supposed to have been first proved by the Greek mathematician **Pythagoras**, who died about 500 B.C. Pythagoras in his later life settled in Italy and was identified with a group of mathematicians and philosophers known as the *Pythagorean School*. The school itself was disrupted after about 200 years, owing to political disturbances; but its influence continued a strong factor in the study and development of mathematics.

197. Corollary 1. *The square on either side of a right triangle is equivalent to the square on the hypotenuse diminished by the square on the other side.*

EXERCISES

1. A baseball diamond is a square 90 ft. on a side. What is the distance from home plate to second base? (Extract square root correct to two decimal places.) *Ans.* 127.27 ft.

2. A ladder 30 ft. long is placed against a wall with its foot 8 ft. from the wall. How far is the top of the ladder from the ground?

3. A tree is broken 20 ft. from the ground. The top strikes the ground 20 ft. from the foot, while the other end of the broken part remains attached to the trunk. How high was the tree?

4. Two columns 60 ft. and 40 ft. high respectively are 30 ft. apart. What is the distance between their summits?

5. Show that it is possible to have a right triangle whose hypotenuse and two sides have the respective values 5, 4, 3. In general, can a right triangle be found whose hypotenuse and two sides have *any* three given values as h, a, b? If not, when is it possible?

6. Find the altitude, and then the area, of an equilateral triangle having a side equal to 6 in.

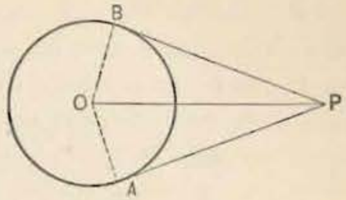
7. Find the area of an equilateral triangle whose altitude is 6 in.

8. Prove the theorem stated in Ex. 11, p. 167, by means § 196.

9. Show that the square of the diameter of a circle is equal to the sum of the squares of any two chords drawn from a point on the circle to the ends of the diameter. (§ 133.)

10. A kite (see Ex. 21, p. 123) is inscribed in a circle whose diameter is 24 ft. If the length of one of the two longer sides of the kite is 18 ft., how long is one of the shorter sides?

11. Show that the sum of the squares of the four sides of *any* kite (Ex. 30, p. 87) is equal to twice the sum of the squares of the four segments of the cross formed by its diagonals.



12. Show that the square of the length of the tangent AP to a circle from a point P plus the square of the radius of the circle is equal to the square of the distance OP from P to the center of the circle.

13. In the figure of Ex. 12, let $OP=p$, $OA=r$, $AP=t$. Show by means of Ex. 12 that

$$t^2=p^2-r^2=(p-r)(p+r).$$

Since $p+r$ is the length of the whole secant from P through O to the opposite side of the circle, and since $p-r$ is the external segment of this secant, show that the preceding equation also results from § 169.

14. In any right triangle ABC (see Fig. 126, p. 151), draw a circle with radius CB about C as center. Using the lettering of Fig. 126, reprove Th. VI by means of § 169, by showing that

$$\begin{aligned} AB^2=AE\cdot AD&=(AC-EC)(AC+CD)\\ &=(AC-CB)(AC+CB)\\ &=AC^2-CB^2,\end{aligned}$$

or

$$AC^2=AB^2+CB^2.$$

198. Definition. The **projection** of a line AB upon another

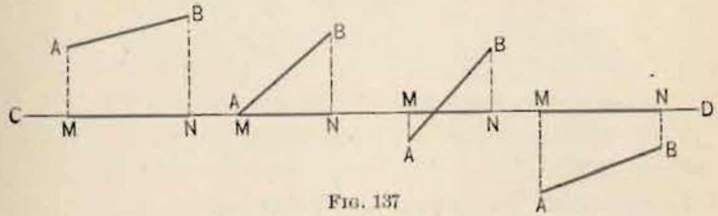


FIG. 137

line CD is the portion of CD cut off between the perpendiculars drawn from the extremities of AB to CD. Thus, in the figure, MN is in each instance the projection of AB upon CD.

199. Theorem VII. *In any triangle the square on the side opposite the* **acute** *angle is equal to the sum of the squares on the other two sides diminished by twice the product of one of those sides and the projection of the other upon it.*

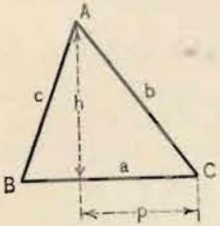


FIG. 138

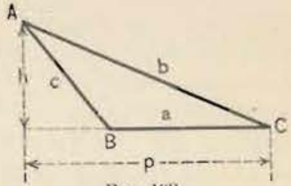


FIG. 139

Given the $\triangle ABC$ in which C is an acute angle. Let a, b, c be the sides opposite the angles A, B, C respectively and let p represent the projection of b upon a.

To prove that $c^2 = a^2 + b^2 - 2\,ap$.

Proof. Draw the altitude h upon the base a. Then, in Fig. 138,

$$c^2 = h^2 + (a-p)^2. \qquad \text{Why?}$$

But $h^2 = b^2 - p^2$; Why?

hence $c^2 = b^2 - p^2 + a^2 - 2\,ap + p^2$, Why?

or $c^2 = a^2 + b^2 - 2\,ap$.

Likewise, in Fig. 139, we have $c^2 = h^2 + (p-a)^2$. The remaining details in this case are left to the student.

EXERCISES

1. Show that if $b = c$ in Fig. 138, $a^2 = 2\,ap$, or $a = 2\,p$. Compare § 43.

2. Show that if $c = a$ in Fig. 138, $b^2 = 2\,ap$.

3. Show, from Ex. 2, that the base of any isosceles triangle is a mean proportional between one of the equal sides and twice the projection of the base upon it.

200. Theorem VIII. *In any obtuse triangle the square on the side opposite the* **obtuse** *angle is equal to the sum of the squares on the other two sides increased by twice the product of one of those sides and the projection of the other upon it.*

Given the obtuse $\triangle ABC$ in which C is the obtuse angle. Let a, b, c be the sides opposite the angles A, B, C respectively and let p represent the projection of b upon a.

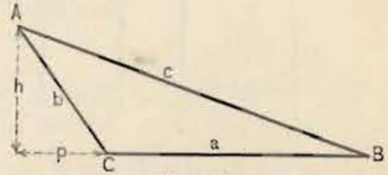


FIG. 140

To prove that $c^2 = a^2 + b^2 + 2\,ap$.

[The proof, being similar to that of § 199, is left to the student.]

EXERCISES

[The symbol * means that the exercise requires the use of trigonometric ratios.]

1. How does the square on any side of a triangle opposite an acute angle compare with the sum of the squares on the other two sides? Answer the same question for the side opposite the obtuse angle in an obtuse angled triangle.

2. Show that either Theorem VII or VIII when applied to a right triangle gives Theorem VI.

3. Find the area of an isosceles triangle whose side is 11 in. and whose base is 8 in.

4. Prove that the sum of the squares on the diagonals of a parallelogram is equal to the sum of the squares on the four sides.

*5. Show that the projection of any line AB on another line CD (Fig. 137) is equal to the product of that line and the cosine of the angle between AB and a parallel to CD through A.

*6. By means of Ex. 5, show that Theorem VII gives $c^2 = a^2 + b^2 - 2\,ab \times (\text{cosine of } C)$. [THE COSINE LAW.]

N

201. Problem 1. *To construct a square whose area shall be equal to the sum of the areas of two given squares.*

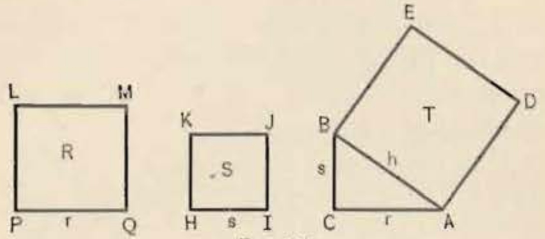


FIG. 141

Given the squares R and S having respectively the sides r and s.

Required to construct a square T whose area shall be the sum of the areas of R and S.

Construction. Draw a right $\triangle$ having r and s as sides.

Upon the hypotenuse h of this triangle construct a square T. This is the square desired.

The proof follows immediately from Theorem VI and is therefore left to the student.

EXERCISES

1. Show how to construct a square equal in area to the sum of three given squares. Generalize your answer to the case of any number of given squares.

2. To construct a square equal in area to the difference of two given squares.

3. Show that a square erected on the diagonal of a given square is equal to twice the given square.

4. Show that the square erected on half the diagonal of a square is equal to half the given square.

5. Construct a square equal to a given triangle.

[HINT. The side of the square is the mean proportional between the base and half the altitude of the triangle. Why?]

202. Problem 2. *To construct a triangle whose area shall be equal to that of a given polygon.*

Given the polygon $ABCDE$.

Required to construct a triangle equivalent to $ABCDE$.

Construction. Draw DA and through E draw $EF \parallel DA$ and meeting BA prolonged at F. Draw DF.

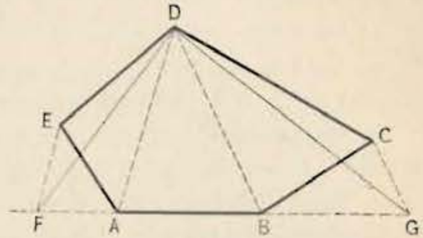


FIG. 142

Then the polygon $FBCD$ has one side less than the polygon $ABCDE$, but is equivalent to it. (See proof below.)

This process may be continued until the last polygon reached is the triangle desired.

Proof. $\triangle AED = \triangle AFD$, since each has the same base AD and their altitudes are equal. § 190, *d*

Adding polygon $ABCD$ to both members of this equation, we obtain Polygon $FBCD =$ Polygon $ABCDE$.

Similarly we have Polygon $FBCD = \triangle FGD$.

Therefore $\triangle FGD$ is the triangle required.

EXERCISES

1. Construct a triangle equivalent to a square whose side is 2 in. Is your triangle the *only* such triangle?

2. Construct a square equivalent to a given parallelogram.

[HINT. The side of the square is the mean proportional between the base and altitude of the parallelogram. Why?]

3. How could a square be constructed that would be equivalent to a given pentagon?

[HINT. Construct a triangle equal to the given pentagon; then proceed as in Ex. 5, p. 178.]

MISCELLANEOUS EXERCISES. CHAPTER IV

[The symbol * means that the exercise involves Trigonometric ratios.]

1. Find the area of a square whose diagonal is 30 ft. long.

2. Find the difference in boundary between a rectangle whose base is 16 ft. and a square equal to it whose side is 12 ft.

3. Prove that the diagonal of a trapezoid divides it into two triangles whose areas are proportional to the bases.

4. Show, by Ex. 3, that the perpendiculars let fall upon one diagonal of a trapezoid from the ends of the other diagonal are proportional to the parallel bases.

5. Show how to find, from proper measurements, the area of a vacant lot bounded by four streets, only one pair of which are parallel.

6. A field of the form of a right triangle containing 9 acres is represented on a map by a right triangle whose sides are 17 in. and 25 in. On what scale is the plan drawn? [1 acre = 4840 sq. yd.]

7. What is the locus of all points such that the sum of the squares of the distances of any one of them from two fixed points is equal to the square of the distance between those two points?

8. Obtain the formula for the area of an isosceles right triangle whose hypotenuse is h.

9. Obtain the formula for the area of an isosceles triangle whose base is b and whose side is a.

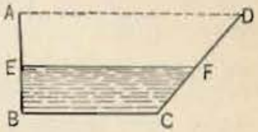
10. From Ex. 9 derive a formula for the area of an equilateral triangle.

11. If one side of a right triangle is three times the other, how long is the hypotenuse as compared with the shorter side?

12. In Ex. 11, what are the lengths of the segments of the hypotenuse made by a perpendicular from the vertex of the right angle, in terms of the smaller side of the triangle?

13. A channel for water has a cross section which is of the form of a trapezoid $ABCD$.

If AD and BC are known and the total height H of the channel is known, find the area of the cross section.



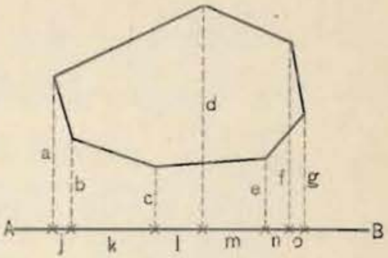
If the depth of the water is h, find the area of the cross section of the water in the channel; (1) when $h = H/2$; (2) when $h = H/4$; when $h = 2\,H/3$.

Find the length of CD in terms of H if the angle ADC is 45°.

If the water level is EF, find CF in terms of h, if the angle ADC is 45°.

14. Do the areas mentioned in Ex. 13 change if BC, H, and AD remain fixed, but the angles at B and C are changed as represented in the figure? Explain your answer.

15. The area of a field may be found in the following way: Run (stake out) a line (base line) AB. From certain points in the boundary of the field the distances to this line are measured, as a, b, c, d, e, f, g, and the distances, j, k, l, m, n, o, are also measured. Complete the description of how to proceed.



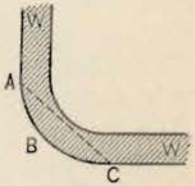
16. To lay off a right angle, carpenters frequently use three sticks 3 ft., 4 ft., and 5 ft. long, respectively. Show that these sides form a right triangle.

17. Show that any multiples of 3, 4, 5, may be used in place of 3, 4, 5 of Ex. 16.

18. Can you discover any three integral numbers a, b, c, not multiples of 3, 4, 5, for which $a^2 + b^2 = c^2$?

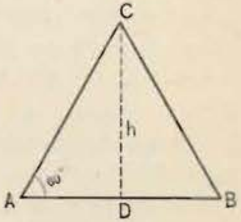
19. Prove that if the middle point of one of the non-parallel sides of a trapezoid is joined to the extremities of the other non-parallel side, the area of the triangle thus formed is equal to half that of the trapezoid.

20. W is a wall having a round corner upon which a gutter is to be placed, and it is desired to find the radius of the circle of which ABC is a quadrant. If the line AC measures 24 ft., show that the desired radius will be about 17 ft.



21. Find the diagonals of a rhombus whose side is 6 ft. 1 in. and whose area is 9 sq. ft.

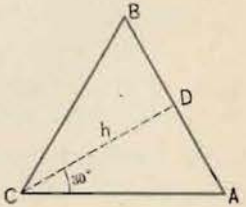
22. Find a formula for the altitude h of an equilateral triangle in terms of one side a, and find the ratio of this altitude to (1) the whole side, (2) one half of one side.



* **23.** From Ex. 22 show that the *sine* of 60° is equal to $\sqrt{3}/2$. Show also that the *cosine* of 60° is equal to $\frac{1}{2}$; and that the *tangent* of 60° is equal to $\sqrt{3}$. (See § 177.) Check by means of the table, p. 155.

* **24.** By means of an isosceles right triangle, show that the sine of 45° and the cosine of 45° are each equal to $\sqrt{2}/2$. Check by the table, p. 155.

* **25.** Turning the figure of Ex. 23 into the new position shown below, find the *sine*, the *cosine*, and the *tangent* of 30°. Check by the table, p. 155.

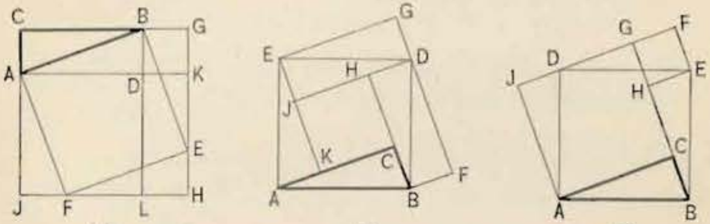


* **26.** Ex. 11 may be restated as follows: if the *tangent* (§ 177) of an angle is 3, what is the *cosine* of that angle? Show, by Ex. 11, that the cosine of the angle is $1/\sqrt{10}$. Check by the table, p. 155.

27. How much leather is required to cover a window seat, in the shape of half of a regular hexagon, if the length of each of its three sides is 5 ft.?

28. Show how to *construct* a line parallel to the bases of a given parallelogram or triangle that shall divide the figure into two parts that are equal in area.

29. Prove the Pythagorean theorem by means of some one of the adjoining figures.



(*a*) (*b*) (*c*)

NOTE. Figure (*a*) is said to be the one used by **Pythagoras**, who discovered the theorem.

Figure (*b*) is due to **Bhaskara**, a native of India, about 1150 A.D.

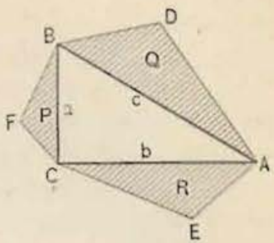
The figure used in § 196 is that used by **Euclid**, who wrote a famous treatise on Geometry about 250–300 B.C.

30. Prove the fact stated in Ex. 9, p. 157, by means of § 189.

31. A room is 18 ft. long, 14 ft. wide, and 10 ft. high. Find the length of one diagonal of the floor. Then find the length of a string stretched through the center of the room from one corner at the floor to the farthest opposite corner at the ceiling. *Ans.* 24.9 ft.

32. If n is any integer, show that $2n$, n^2-1, and $n+1$ are integers that are proportional to the three sides of a certain right triangle. Find these three integers when $n=2$; when $n=4$.

33. Any polygon described on the hypotenuse of any right triangle as one side is equal to the sum of the two similar polygons drawn on the sides, respectively, with the corresponding side as one side of the polygon. Prove this statement.



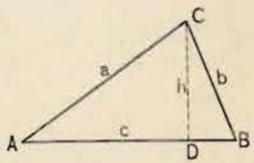
[Hint. Suppose the similar polygons are triangles, as in the figure. Then

$P/Q = a^2/c^2$, $R/Q = b^2/c^2$; hence $Q = P + R$.]

34. Show how to construct the side of an equilateral triangle whose area is the sum of two given equilateral triangles.

35. In any triangle ABC let us write $s = (a + b + c)/2$, where a, b, c, are the three sides. Show that

Area $\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}$.



Outline of Proof. Let A be an acute angle, and let h be the altitude from C; then we have

$$h^2 = a^2 - \overline{AD}^2,$$

but $$b^2 = a^2 + c^2 - 2c \cdot AD, \text{ by § 199;}$$

hence $$h^2 = a^2 - \left[\frac{a^2 + c^2 - b^2}{2c}\right]^2 = \frac{4a^2c^2 - (a^2 + c^2 - b^2)^2}{4c^2}$$

$$= [2ac + (a^2 + c^2 - b^2)][2ac - (a^2 + c^2 - b^2)] \div 4c^2.$$

But $$2ac + (a^2 + c^2 - b^2) = (a^2 + 2ac + c^2) - b^2$$

$$= (a + c)^2 - b^2 = (a + c + b)(a + c - b)$$

$$= 4\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2} - b\right) = 4s(s-b).$$

Likewise

$$2ac - (a^2 + c^2 - b^2) = b^2 - (a - c)^2 = 4(s - c)(s - a).$$

Hence $$h^2 = 4s(s-a)(s-b)(s-c) \div c^2,$$

and $$\text{Area } \triangle ABC = h \cdot c/2 = \sqrt{s(s-a)(s-b)(s-c)}.$$

36. Find the area of a triangular field whose sides are, respectively, 80 rd., 220 rd., 200 rd. [1 acre = 160 sq. rd.]

37. Find the area of a parallelogram whose sides are, respectively, 8 in. and 10 in. long, and one of whose diagonals is 15 in. long. *Ans.* 74.0 sq. in.

38. Find the area of a triangle, given two sides 15 ft. and 25 ft., and their included angle 30°.

* 39. Find the area of a triangle, given two sides 15 ft. and 25 ft., and their included angle 40°. *Ans.* 120.5 ft.

40. Show that the area of any triangle is $s \cdot r$, where $s = (a + b + c)/2$, as in Ex. 35, and r is the radius of the inscribed circle. (See § 124.)

41. Show that the area of any polygon circumscribed about a circle is half the sum of its sides times the radius of the circle.

42. Comparing the two results of Exs. 35 and 40, show that the radius of a circle inscribed in any triangle is

$$r = \sqrt{(s-a)(s-b)(s-c)/s}.$$

43. Show that in any triangle one of whose angles is 120°, the square on the side opposite the larger angle equals the sum of the squares on the other two sides plus the product of those sides.

44. Show that in any triangle one of whose angles is 60° the square on the side opposite that angle is equal to the sum of the squares on the other two sides diminished by the product of these two sides.

* 45. Show, in Figs. 138 and 139, that $h = b \times (\text{sine of } \angle C)$; hence show that the area of the triangle ABC is

$$\text{Area } \triangle ABC = \tfrac{1}{2}ab \times (\text{sine of } \angle C).$$

* 46. Prove Theorem IV, § 193, by means of Ex. 45.

CHAPTER V

REGULAR POLYGONS AND CIRCLES

203. Regular Polygon. A polygon that is both equiangular and equilateral is called a **regular** polygon. (See § 96.)

EXERCISES

1. Construct a quadrilateral that is equilateral but not equiangular. What are such quadrilaterals called?

2. Construct a quadrilateral that is equiangular but not equilateral. What are such quadrilaterals called?

3. Draw a quadrilateral that is neither equiangular nor equilateral.

4. Construct a regular polygon of four sides. What are such quadrilaterals called?

5. Is an equilateral triangle a regular polygon? Why?

6. What is the size of each angle of a regular polygon of seven sides? (See § 97.)

7. If three equal rods are hinged together at their ends in the form of an equilateral triangle, is the framework rigid?

8. If four equal rods are joined together at their ends by hinges, is the framework formed necessarily a square? Can the angles be changed without changing the lengths of the sides?

9. If five equal rods are hinged together at their ends in the form of a regular pentagon, is the framework rigid?

10. Will the framework of Ex. 8 be rigid if a stiff rod is inserted along one diagonal? Along how many diagonals must rods be inserted in the framework of Ex. 9 to make it rigid?

204. Theorem I. *If a circle is divided into a number of equal arcs:*

(a) the chords joining the points of division form a regular inscribed polygon;

(b) tangents drawn at the points of division form a regular circumscribed polygon.

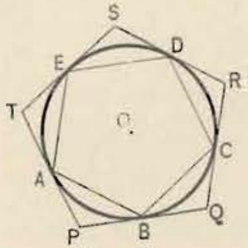


FIG. 143

Given the circle divided into the equal arcs AB, BC, CD, DE, and EA by the chords AB, BC, CD, DE, and EA; and given the tangents PQ, QR, RS, ST, and TP, touching the circumference at the points B, C, D, E, and A, respectively.

To prove that (*a*) $ABCDE$ is a regular polygon;
(*b*) $PQRST$ is a regular polygon.

Proof of (*a*). Since $\widehat{AB}=\widehat{BC}=\widehat{CD}=\widehat{DE}=\widehat{ED}$,		Given
Arcs ABC, BCD, CDE, DEA, and EAB are equal.		Why?
Hence	$\angle ABC = \angle BCD = \angle CDE$, etc.	Why?
Also	$AB = BC = CD = DE$, etc.	Why?
It follows that $ABCDE$ is a regular polygon.		§ 203

Proof of (*b*). Show that $\triangle APB \cong \triangle BQC$ by showing that $AB = BC$, $\angle PAB = \angle QBC$ (§ 137), and $\angle PBA = \angle QCB$. Likewise $\triangle APB \cong \triangle CRD$, etc. Then show that $PQ = QR$ = etc.; and $\angle P = \angle Q$ = etc.; and thus prove the polygon $PQRST$ a regular polygon.

205. Theorem II. *(a) A circle may be circumscribed about any regular polygon;*

(b) a circle may also be inscribed in it.

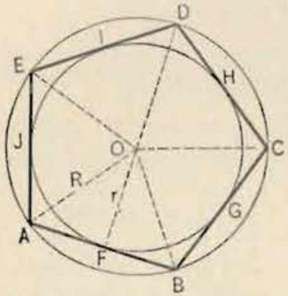


FIG. 144

Given the regular polygon $ABCDE$.

To prove that (*a*) a circle may be circumscribed about it;
(*b*) a circle may be inscribed in it.

Proof of (*a*). Pass a circle through the points A, B, and C. § 120

Join the center O with the vertices of the polygon.

Then	$OB = OC$, and $AB = CD$.	§§ 103, 203
Moreover	$\angle CBA = \angle DCB$,	§ 203
and	$\angle CBO = \angle OCB$.	§ 40
Therefore	$\angle OBA = \angle OCD$,	Ax. 2
so that	$\triangle OAB \cong \triangle OCD$;	§ 35

hence $OA = OD$, and the circle passes through D. Why?

In like manner show that the circle passes through E.

Hence the circle is circumscribed about the polygon. § 114

Proof of (*b*). The sides of the regular polygon are equal chords of the circumscribed circle. Therefore they are equally distant from the center. § 109

The circle with O as a center and the perpendicular from the center to any side as radius is inscribed in the polygon. § 114

206. Definitions. The common center of the circumscribed and inscribed circles is called the **center** *of the regular polygon.*

The radius R of the circumscribed circle is called the **radius** *of the regular polygon.*

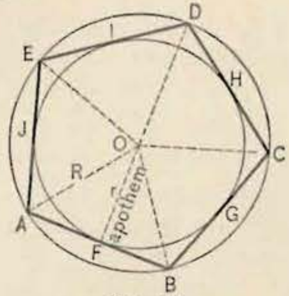


FIG. 145

The radius r of the inscribed circle is called the **apothem** *of the regular polygon.*

The **perimeter** of a polygon is the sum of the lengths of its sides.

207. Theorem III. *The area of a regular polygon is equal to half the product of its apothem and perimeter.*

Given the regular polygon $ABCDE \cdots$, Fig. 145. Let its perimeter be denoted by p and its apothem by r.

To prove that $ABCD \cdots = pr/2$.

Proof. Draw the radii OA, OB, OC, etc., Fig. 145, thus making as many triangles as the polygon has sides.

The altitude of each triangle is the apothem r, and each base is a side of the polygon; hence, the area of each triangle is half the product of r and one side of the polygon. § 189

Since the sum of all the bases is the perimeter p of the polygon, and the sum of the areas of the triangles is the area of the polygon, it follows that

$$\text{Area of } ABCD \cdots = pr/2.$$

Compare this result with Ex. 41, p. 185.

EXERCISES

1. If n represents the number of sides of a regular polygon, show that the angle at the center subtended by one side is $4/n$ right angles.

2. Find the angle at the center subtended by one side of an equilateral triangle; of a square; of a regular pentagon; of a regular hexagon. Draw these figures by means of a protractor.

3. Prove that any angle of a regular polygon is supplementary to the angle at the center subtended by one side.

4. Find the number of degrees in the angle at the center subtended by one side of a regular octagon. Find the size of one angle of the octagon by means of Ex. 3; and then check your answer by § 97. Draw the figure.

5. How many sides has a regular polygon if the angle at the center subtended by one side is 40°?

6. If one angle of a regular polygon is 140°, what is the number of its sides?

7. Find the area of a regular hexagon whose side is 6 in.

8. Two squares have for their sides 8 and 11 in., respectively. Find by § 207 the ratio of their areas.

9. Two squares have areas of 144 sq. in. and 225 sq. in., respectively. What is the ratio of their perimeters?

10. Find the ratio of the perimeters of squares inscribed in and circumscribed about the same circle. *Ans.* $1 : \sqrt{2}$.

11. The perimeter of a regular inscribed hexagon is 30 in. Find the perimeter of a regular hexagon circumscribed about a circle of twice the diameter. *Ans.* 69.28 in.

12. Find the area of an equilateral triangle circumscribed about a circle whose radius is 12 in. *Ans.* 748.25 sq. in.

13. Find the apothem of an equilateral triangle of which one side is 6 in. (Use Th. XXXIII, § 102.) Hence find its area by Theorem III. Compare with the result of Ex. 6, p. 174.

208. Areas of Inscribed and Circumscribed Regular Polygons. The area of a circle is evidently greater than the area of any regular polygon inscribed in it. (17, § 31.)

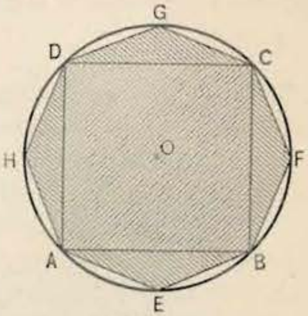


FIG. 146

By doubling the number of sides, we obtain regular inscribed polygons whose areas are more nearly equal to that of the circle, as indicated in the figure.

Thus the area of a regular inscribed octagon, for example, is evidently more nearly equal to the area of the circle than is the area of an inscribed square.

The area of a circle is also evidently less than the area of any circumscribed polygon.

209. Perimeters. The perimeter of any inscribed polygon is evidently less than the length of the circumference (Post. 3, § 28); we shall also assume that the length of the circumference is less than that of any circumscribed polygon.

210. Areas and Lengths of Circles. Since the regular inscribed and regular circumscribed polygons approach, both in length and in area, nearer to one another and to the circle, as the number of sides is doubled, we may say:

If the number of sides of the regular inscribed and regular circumscribed polygons is repeatedly doubled,

(*a*) *their areas approach the area of the circle as a common limit;*

(*b*) *their perimeters approach the length of the circumference of the circle as a common limit.*

It is also obvious that the apothem of the inscribed regular polygon will approach the radius of the circle, under the same circumstances.

211. Theorem IV. *The circumferences of two circles are to each other as their radii.*

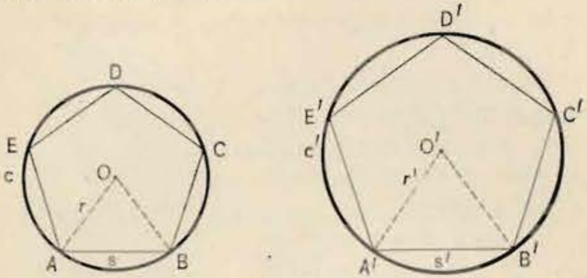


FIG. 147

Given two circles O and O' whose radii are r and r'.

To prove that the length of their circumferences c and c' are to each other as their radii; that is,

$$c/c' = r/r'.$$

Proof. Draw inscribed regular polygons P and P', of the same number of sides in each circle, let the perimeters of these polygons be p and p', and let the corresponding sides be s and s'.

Then $\quad P \sim P'$, $\quad$ § 166

and $\quad p/p' = s/s'$. $\quad$ § 167

But $\quad s/s' = r/r'$,

since the triangle two of whose sides are s and r is similar to the triangle two of whose sides are s' and r'.

Hence $\quad p/p' = r/r'$. $\quad$ Ax. 9

Since p and p' approach c and c' if the number of sides is repeatedly doubled, the difference $p/p' - c/c'$ can be made as small as we please by doubling the number of sides repeatedly; hence also $r/r' - c/c'$ can be made as small as we please.

But r/r' and c/c' do not change at all as we increase the number of sides of the polygons; hence

$$r/r' = c/c',$$

for if r/r' and c/c' differed from each other, their difference would be fixed, and therefore not as small as we might please.

212. Corollary 1. *The ratio of a circumference to its diameter is the same for all circles.*

For, since $c/c' = r/r'$ (§ 211), we have also, if d and d' are the diameters, $d/d' = 2\,r/2\,r' = r/r' = c/c'$.

Hence by *alternation* (Theorem D, § 144), $c/d = c'/d'$.

213. The Number π. *The number obtained by dividing the circumference of any circle by its diameter,* which, by § 212, is the same for all circles, *is denoted by the Greek letter π* (pronounced pi). This number, which is about $3\frac{1}{7}$ (or, more accurately still, 3.1416), will be computed approximately later in this chapter.

214. Corollary 2. *In any circle $c = \pi d$, or $c = 2\,\pi r$, where r is the radius, d the diameter, and c the length of the circumference.*

For, since $\pi = \dfrac{c}{d}$ (§ 213), $c = \pi d$; or since $d = 2\,r$, $c = 2\,\pi r$.

EXERCISES

1. Find the length of the circumference of a circle of radius 2 ft. [Take $\pi = 3\frac{1}{7}$.] *Ans.* $4\,\pi$ ft., or 12.57 ft.

2. If the radius R of the earth is 4000 mi., what is its circumference at the equator? [Take $\pi = 3\frac{1}{7}$.] How much is this result affected by taking the more accurate values $R = 3963$ mi., $\pi = 3.1416$.

3. Find the diameter of a circle whose circumference is 40 in.

4. Measure the circumference of some round object, such as a porch column, by stretching a string around it tightly and then measuring the string. Now compute the diameter.

5. How wide must a piece of tin be cut in order to be made into a stovepipe 8 in. in diameter? *Ans.* $8\,\pi$ in., or $25\frac{1}{7}$ in.

6. Show how to find quickly the approximate length of wire in a coil by measuring the diameter of the coil, and counting the number of strands.

7. What is the length of an arc that subtends an angle of 60° at the center of a circle whose radius is 5 ft.?

215. Theorem V. *The area of a circle is equal to one half the product of its radius and its circumference.*

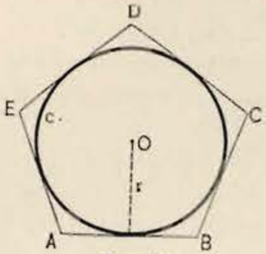


FIG. 148

Given the circle O, with radius r, circumference c, and area A.

To prove that $A = rc/2$.

Proof. Circumscribe a regular polygon about the circle. Let A' denote its area and p' its perimeter.

Then $A' = rp'/2$. § 207

As the number of sides of the regular circumscribed polygon is increased, p' approaches c as its limit. § 210

Hence $rp'/2$ approaches $rc/2$ as a limit.

Also A' approaches A as a limit. § 210

Therefore the difference between A and $rc/2$ must be as small as we please, as in § 211. It follows, as in § 211, that $A = rc/2$.

The student should state carefully all of the remaining steps in the argument.

216. Corollary 1. *The area of a circle is equal to π times the square of its radius, that is, $A = \pi r^2$.*

For, by § 215, $A = rc/2$; but $c = 2\pi r$; hence $A = \pi r^2$.

217. Corollary 2. *The areas of two circles are to each other as the squares of their radii.*

NOTE. The very famous problem of "squaring the circle," that is, of constructing the side of a square whose area equals that of a given circle, depends on determining the value of π. We now know that this construction is impossible with ruler and compasses; but the ancient Greeks and the Schoolmen of the Middle Ages spent much time attempting to do it.

218. Sectors. The area of a sector bears the same ratio to the area of a circle as the angle of the sector bears to 360°. For example, the area of a sector whose arc is 36° is 1/10 the area of the circle.

219. Circle divided into Sectors. A circle may be cut into equal sectors and arranged as in the following figure.

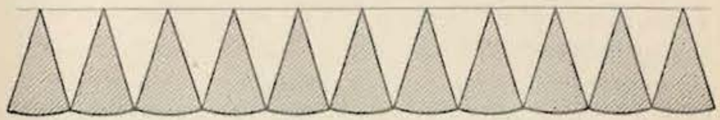


FIG. 149. Circle divided into Sectors

Each sector approximates the shape of a triangle. The area of each sector is equal to one half the product of its arc (base) by its radius (altitude). Their sum equals one half the whole circumference times the radius.

EXERCISES

[Exercises that involve the number π should be worked through first in terms of the symbol π; then the value $3\frac{1}{7}$ should be substituted for it. Thus the area of $\frac{1}{4}$ of a circle whose radius is 6 inches is $\frac{1}{4} \cdot 6^2 \cdot \pi$ sq. in. $= 9\pi$ sq. in.; substituting $\pi = 3\frac{1}{7}$, we find the result $28\frac{2}{7}$ sq. in.]

1. Find the area of a circle whose diameter is 14 in.

2. Find the area of a square circumscribed about a circle whose radius is 2 ft. What is the ratio of the area of the circle to the area of this square? *Ans.* 16 sq. ft.; $\pi/4$, or 11/14.

3. The circumference of a circle is 14 ft. Find its radius and its area.

4. The area of a circle is 24 sq. in. Find its radius.

5. What is the change in the area of a circle, if its radius is multiplied by 2? by 5? by n?

6. The effect of an explosion in a large powder plant was felt for a distance of 10 mi. in every direction. How large an area was affected?

7. A tinner is to cut the largest possible square from a circular piece of tin. What proportion of the tin will be left?

8. Find the area of a sector in a circle of radius 3 ft. if the angle of the sector is 45°; 60°; 30°; 80°.

9. If a tinner cuts from the same sheet of tin two circular pieces, one of which is twice as wide as the other, how much heavier is the larger one?

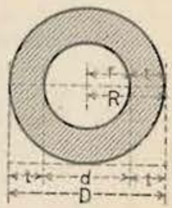
10. If it is desired to cut two circular weights out of a flat piece of metal, how much wider must the larger be to weigh twice as much, the thickness being the same?

11. A steel rod whose cross-section is 1 in. square weighs 3.4 lb. per foot of length. Find the weight per foot of length of a round rod 2 in. thick of the same material.

12. Find the weight per foot of length of a waterpipe whose outer diameter is 3 in., and whose inner diameter is 2.75 in., made of the material mentioned in Ex. 11.

13. Show that the area of a circular ring contained between two concentric circles of diameters D and d is

$$A = \pi\left(\frac{D^2 - d^2}{4}\right), \text{ or } A = \pi\frac{D-d}{2}\cdot\frac{D+d}{2}.$$



14. How much water per hour (in cubic feet) will flow through a pipe whose inner diameter is 3 in. if the water is flowing 3 ft. per second?

220. Problem 1. *Given the side and radius of a regular inscribed polygon, to find the side of a regular inscribed polygon of double the number of sides.*

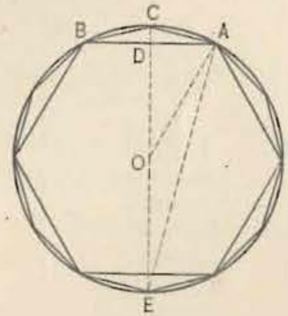


FIG. 150

Given AB, the side of a regular inscribed polygon of radius r.

Required to find AC, a side of the regular inscribed polygon of double the number of sides.

Solution. Draw the diameter CE, and the radius AO. Also draw AE.

Now $OD \perp AB$ at its middle point. Why?

Hence $\overline{OD}^2 = r^2 - \frac{1}{4}\overline{AB}^2$, Why?

or $OD = \sqrt{r^2 - \frac{1}{4}\overline{AB}^2}$,

and $CD = r - \sqrt{r^2 - \frac{1}{4}\overline{AB}^2}$.

Also $\overline{AC}^2 = CE \cdot CD = 2r \cdot CD$; § 163

that is,

$$AC = \sqrt{2r\left(r - \sqrt{r^2 - \frac{1}{4}\overline{AB}^2}\right)}$$

$$= \sqrt{r\left(2r - \sqrt{4r^2 - \overline{AB}^2}\right)}.$$

221. Corollary 1. *If $r = 1$, and s = the side of the inscribed polygon,*

$$AC = \sqrt{2 - \sqrt{4 - s^2}}.$$

222. Problem 2. *To compute approximately the value of π.*

The perimeter of a regular hexagon inscribed in a circle of unit radius is 6 units. (Why?) By using the formula in § 221 and computing successively the perimeters for polygons of 12, 24, 52, · · ·, and 768 sides we get the following results:

NUMBER OF SIDES	LENGTH OF ONE SIDE	LENGTH OF PERIMETER
12	.51763809	6.21165708
24	.26105238	6.26525722
48	.13080626	6.27870041
96	.06543817	6.28206396
192	.03272346	6.28290510
384	.01636228	6.28311544
768	.00818126	6.28316941

By continuing this process it is found that the first five figures in the decimal remain unchanged. Hence 6.28317 is a close approximation to the circumference of a circle whose radius is 1. Since the diameter is 2, the ratio π of the circumference to the diameter is, approximately,

$$\pi = \frac{6.28317}{2} = 3.14159 \text{ (usually written 3.1416, or } 3\tfrac{1}{7}\text{)}.$$

NOTE. A still more accurate value can be computed by continuing the preceding process. It has been proved that the number π cannot be expressed precisely by any finite decimal.

The fact that π cannot be expressed precisely is equivalent to the statement that *the diameter and the circumference of a circle are incommensurable to each other* (§ 128). But we can obtain, by the preceding process, as great accuracy as we please.

The value is known to over 700 decimal places. To ten places it is $\pi = 3.1415926536$, but such accuracy is never necessary in any ordinary affairs. As a curiosity, we quote the value:

$\pi = 3.14159265358979323846264338327950288419716939937 51.$

223. Problem 3. *To inscribe a square in a given circle.*

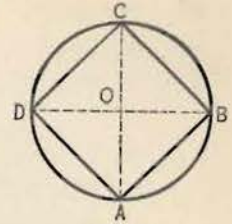


FIG. 151

Given the circle O.

To inscribe a square in circle O.

Construction. Draw two diameters AC and DB perpendicular to each other.

Draw AB, BC, CD, and DA.

Then $ABCD$ is the square desired.

Proof. [The proof is left for the student.]

224. Problem 4. *To inscribe a regular hexagon in a given circle.*

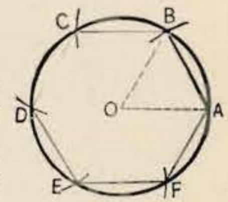


FIG. 152

Given the circle O.

To inscribe a regular hexagon in circle O.

Construction. Draw the radius OA and with A as a center and radius OA draw an arc cutting the circle in B. Then AB is the side of the hexagon desired.

Outline of Proof. Draw OB and show that $\widehat{AB}$ subtends a central angle 60°.

EXERCISES

1. Show how to inscribe an equilateral triangle.
2. Show how to inscribe a regular polygon of twelve sides.
3. Since one side of an inscribed square subtends a central angle 90°, and one side of a regular inscribed hexagon subtends a central angle of 60°, show that if one vertex of the square coincides with one vertex of the hexagon, the next vertices are at the extremities of an arc of 30°.
4. Show how to inscribe a regular polygon of eight sides.
5. Show how to inscribe a regular polygon of twenty-four sides directly from an inscribed regular octagon and a regular inscribed hexagon.

225. Problem 5. *To inscribe a regular decagon.*

Given the circle O.

Required to inscribe a regular decagon in the given circle.

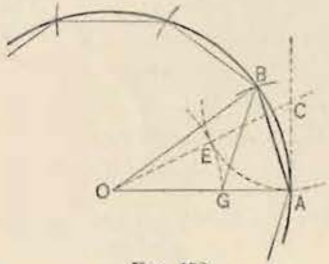


FIG. 153

Construction. Draw the radius OA and divide it in extreme and mean ratio at G, having the larger segment next to the center. § 172

Then $OA : OG = OG : GA$, and OG is the side of the decagon required. By applying OG ten times to the circle as a chord, the desired regular decagon is formed.

Proof. Draw OB and BG.

Since $OA/OG = OG/GA$ and $OG = AB$, Const.

we have $OA/AB = AB/GA$; Why?

hence $\triangle OAB \sim \triangle ABG$. Why?

Therefore $\angle O = \angle ABG$ and $\angle AGB = \angle ABO$. Why?

But $\angle OAB = \angle ABO$; Why?

hence $\angle AGB = \angle GAB$, and $AB = BG = GO$. Why?

Therefore $\angle O = \angle GBO$. Why?

But we have shown $\angle O = \angle ABG$;

hence, adding, $2 \angle O = \angle OBA = \angle OAB$.

Now $\angle O + \angle OBA + \angle OAB = 2$ rt. $\angle$s, Why?

whence $5 \angle O = 2$ rt. $\angle$s, Why?

or $\angle O = 1/5$ of 2 rt. $\angle$s, or $1/10$ of 4 rt. $\angle$s.

Therefore the arc AB is $1/10$ of the circumference, and the chord AB, equal to OG, is a side of a regular inscribed decagon.

EXERCISES

1. Show how to inscribe a regular pentagon in a circle.

2. Show how to inscribe regular polygons of 20, 40, etc. sides in a given circle.

3. Show that if one vertex of a regular inscribed pentagon coincides with one of a regular hexagon inscribed in the same circle, the next vertices are extremities of an arc of 12°. Hence show how to inscribe a regular polygon of 30 sides.

4. Show how to inscribe a regular polygon of 15 sides directly by using the regular inscribed pentagon and the inscribed equilateral triangle.

MISCELLANEOUS EXERCISES. CHAPTER V

1. The radius of a circle is 2 in. Find the length of the circumference; the area.

2. The area of a circle is 98 sq. ft. Find the diameter.

3. The diameters of two circles are 4 ft. and 9 ft., respectively. Find the ratio of their areas.

4. How many people can be seated at a round table 54 in. in diameter when it is extended 4 ft., allowing 2 ft. to a person?

5. Find the perimeter of a regular hexagon inscribed in a circle whose radius is 1 ft.; 3 ft.; a ft.

6. The perimeter of a regular inscribed hexagon is 48 ft. What is the diameter of the circle?

7. Find the perimeter of a regular circumscribed hexagon, if the radius of the circle is 1 ft.; 3 ft.; r ft.

8. If the radius of a circle is r, find the area of the inscribed equilateral triangle; of the circumscribed equilateral triangle.

9. Show that the area of the inscribed equilateral triangle equals one fourth the area of the circumscribed equilateral triangle.

10. A cow is tethered at the end of a 50 ft. rope, which is fastened to the corner of a barn. The barn is 25 ft. wide and 60 ft. long. Over how much area may the cow graze?

11. How many revolutions per mile does a 28-in. bicycle wheel make? *Ans.* 720.

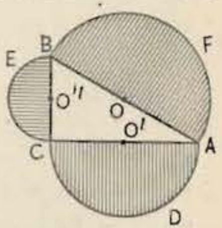
12. The boiler of an engine has 200 tubes, each 3 in. in diameter, for conducting the heat through the water. Find their total cross sectional area.

13. Construct a regular inscribed pentagon, and draw all the diagonals. Show that the sum of all the angles in the vertices of the resulting five-pointed star equals two right angles.

14. A circular piece of brass has a radius of 10 in. and it is desired to cut a hole through it equal in area to one half the disk. What should be the radius of the hole?

15. The central angle whose arc is equal to the radius of the circle is called a **radian**. It is often used as a unit of measure of angles. Show that 1 radian $= 180° \div \pi = 57.3°$ approximately.

16. A circle is circumscribed about a right triangle and two others are described with the sides as diameters. Prove that the large circle equals the sum of the two small ones.



17. Semicircles are constructed on the three sides of a right triangle as in the adjacent figure. Show that the sum of the semicircles $AO'CD$ and $BO''CE$ is equal to the semicircle $AOBF$.

18. Construct a circle equal to the area of two given circles.

19. If the limit of safety for the surface speed of an emery stone is 5500 ft. per minute, what is the diameter of the largest wheel that can safely make 1500 revolutions per minute? *Ans.* $11/(3\pi)$ ft., or 14 in.

20. Can a piece of paper 6 in. wide be used to wrap up a circular mailing roll whose radius is 1 in.?

21. How many laps around a circular running track whose diameter is 125 yd. are necessary to make up a distance of 8 mi.?

22. Find the length of the curved portion of a railroad track that connects two straight portions at right angles to each other, given that the curved portion is an arc of a circle of radius 200 ft., tangent at its extremities to the straight portions of the track.

Find the length of the curved portion if the angle between the straight portions is 60°.

23. What is the total pressure on the piston of an engine if the cylinder is 20 in. in diameter and the gauge shows 76 lb. per square inch?

24. Four pumps each with a diameter of 5 in. are used in a mine. If one pump were used to remove the same amount of water in the same time, what would be its diameter, everything else being the same? *Ans.* 10 in.

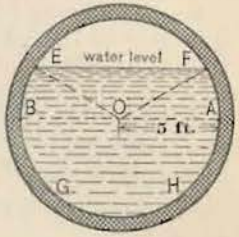
25. When the gauge shows a steam pressure of 100 lb. per square inch, what is the total pressure tending to blow the cylinder head out, if it is 18 in. in inside diameter?

26. If the water in a 3-in. (inside diameter) water main is flowing at the rate of 5 ft. per second, how much water is passing a given point per minute? 1 gal. $= 231$ cu. in.

27. A water main of 6-in. diameter is continued beyond a certain point by a pipe of 4-in. diameter. If the water in the

6-in. pipe is running at the rate of 2 ft. per second, how fast is the water in the 4-in. pipe running?

28. A conduit for carrying water is circular in form and is 10 ft. in diameter. Find the area of the total cross section. If the water level is at EF, find the area of the cross section of the water if the angle EOF is 90°; if EOF is 60°.



29. Find the amount of water flowing through the conduit of Ex. 28 per minute if $\angle EOF = 60°$ and the speed is 2 ft. per second.

30. Find the length $EGHF$ of the portion of the circular outline, Ex. 28, which is wet when the water reaches EF if EOF is 45°; if EOF is 60°; if EOF is equal to 90°. (This so-called *"wetted perimeter"* is of the greatest importance in determining friction and therefore the resistance of the pipe to the water flow.)

* 31. Find the area of the cross section of the water in the conduit of Ex. 28 when $\angle EOF = 80°$. *Ans.* 73.41 sq. ft.

* 32. Find the area of the cross section of the water in the conduit of Ex. 28 when the distance from O to EF is 4 ft.

33. How fast is the point on the rim of a wheel moving, in feet per second, if the wheel is 2 ft. in diameter and is rotating at an angular speed of four revolutions per minute?

34. How far does a carriage move when one of its wheels revolves (without slipping) through five complete revolutions, if the diameter of the wheel is 4 ft.?

35. How many times will the wheel of a bicycle revolve, if it is 28 in. in diameter, in going 3 mi.? If the bicyclist goes 3 mi. in 20 min., how many revolutions does the wheel make per minute?

36. Find the angular speed of a car wheel that is 20 in. in diameter, when the train is going 40 mi. per hour.

37. The radius of the earth is approximately 4000 mi., and the earth makes one revolution per day. What is the speed, in miles per hour, of a point on the equator? of a point whose latitude is 30°?

38. Compare the speed of a point on the equator of the earth, in miles per hour, with the speed of an express train going 60 mi. per hour. Compare it with the speed of a point on the rim of a flywheel 2 ft. in diameter that is making 100 revolutions per second.

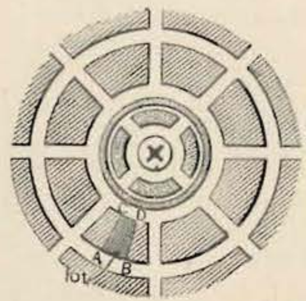
39. If a hollow pipe has an inside diameter d and an outside diameter D, the thickness of its walls t is equal to $(D-d)/2$. Show that the area of the cross section of the metal is

$$A = \pi\,\frac{D+d}{2}\cdot t.$$

Show that $D + d = 2(D - t)$; hence show that $A = \pi(Dt - t^2)$.

40. Show that the area bounded by two concentric circles of radii r and R and two radii of the larger one, is equal to half the product of its *altitude* $(R-r)$ and the sum of its two circular sides. (Compare § 191.)

41. Show how to find the area of a city lot bounded by two circular streets that have a common center, and two of their



radii, if the lengths of the circular arcs are 120 ft. and 160 ft., and the straight line boundaries are 75 ft. long.

APPENDIX TO PLANE GEOMETRY

MAXIMA AND MINIMA

226. Definitions. Let P be a fixed point upon the circumference of a given circle and let a series of chords be drawn through this point. The *longest* one of all these chords is the diameter PQ (Ex. 2, p. 91).

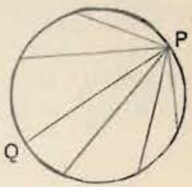


FIG. 154

This fact may be briefly stated by saying:

Of all chords through P, the diameter PQ is the **maximum** (*greatest*).

Again, of all regular polygons that can be inscribed in a given circle, the one whose area is *least* is the inscribed equilateral triangle, a fact which may be stated by saying: *Of all regular polygons inscribed in a circle, the equilateral triangle has the* **minimum** (*least*) *area, or simply, is the* **minimum.**

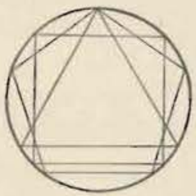


FIG. 155

Likewise, *of all the straight lines that can be drawn from a fixed point to a given line, the perpendicular is the* **minimum** (§ 77).

These and all other similar considerations constitute the subject of **maxima** and **minima** in Geometry. The **maximum** of several quantities is the greatest among them; the **minimum** is the least among them. We shall now state and prove a number of theorems related to this subject.

227. Theorem I. *Of all triangles that have the same two given sides, that in which these sides include a right angle is the maximum.*

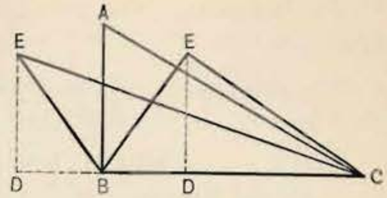


FIG. 156

Given the right $\triangle ABC$ and any other $\triangle EBC$ constructed upon the side BC and having its side $EB = AB$.

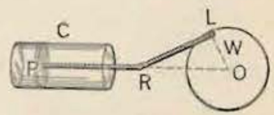
To prove	$\triangle ABC > \triangle EBC.$	
Proof. Draw	$ED \perp BC.$	
Then	$EB > ED.$	§ 75
But	$EB = AB.$	Given
Therefore	$AB > ED.$	Why?
Whence also	$\triangle ABC > \triangle EBC.$	(b), § 190

EXERCISES

1. What is the maximum line that can be drawn within a rectangle and terminated by the sides? What is the minimum line through a given interior point?

2. Draw a circle and take any point P within it. Construct the maximum line and also the minimum line from P to the circumference. Repeat, using a point P outside the circle.

3. In the figure, C is the cylinder of a steam engine, P the piston, PR the piston rod, RL the connecting rod, and W the driving wheel. As the engine works, describe the position of the connecting rod when the area of the triangle RLO is a maximum. (O represents the center of the wheel W.) Answer the same question for a minimum triangle RLO.



228. Definition. Figures having equal perimeters are called **isoperimetric.**

229. Theorem II. *Of all isoperimetric triangles having the same base, the isosceles is the maximum.*

Given the isosceles $\triangle ABC$ and any other $\triangle DBC$ constructed upon BC and isoperimetric to ABC.

To prove $\triangle ABC > \triangle DBC$.

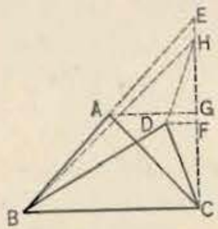


FIG. 157

Proof. Prolong BA to E, making $AE = BA$, and draw EC.

A circle having A as center and BE as diameter can now be drawn through the points B, C, and E. Why?

Therefore $\angle BCE$ is a right angle. Why?

Now draw AG and $DF \parallel BC$

also draw $DH = DC$, and join B and H.

Then	$AB + AC = DB + DC = DB + DH = BE$.	Given
But	$DB + DH > BH$.	Why?
Therefore	$BE > BH$.	
Whence also	$CE > CH$.	Why?
But	$CG = \frac{1}{2} CE$ and $CF = \frac{1}{2} CH$.	Why?
Therefore	$CG > CF$;	Why?
and hence	$\triangle ABC > \triangle DBC$.	Why?

230. Corollary 1. *Of all isoperimetric triangles, the equilateral is the maximum.*

231. Theorem III. *Of all isoperimetric polygons having the same number of sides, the maximum is the one that is equilateral.*

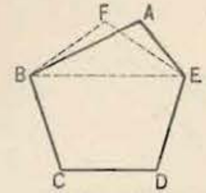


FIG. 158

Given the maximum polygon $ABCDE$ of all those that can be drawn isoperimetric to each other, having the same number of sides.

To prove that $ABCDE$ is equilateral.

Proof. If $ABCDE$ is *not* equilateral, at least two of its sides, as AB and AE, must be unequal; and we may construct on the diagonal BE an isosceles $\triangle BFE$ which is isoperimetric with $\triangle ABE$.

Then	$\triangle BFE > \triangle ABE$.	§ 229
Therefore	$BCDEF > ABCDE$.	

But this is contrary to hypothesis.

Therefore, $AB = AE$ and $ABCDE$ is equilateral.

EXERCISES

1. Show that of all triangles inscribed in a semicircle, that is greatest which has the diameter as base and the radius, r, as altitude. Its area is equal to r^2.

2. Compare the area of the triangle of Ex. 1 with that of the semicircle; with the area of a square inscribed in the circle.

3. Show that of all isoperimetric quadrilaterals, the maximum is a rhombus; show furthermore that it is a square.

4. Prove that of all parallelograms having given sides, the rectangle is the maximum. What does this theorem become when stated with reference to jointed frames? (See Ex. 7, p. 165.)

P

232. Theorem IV. *Of all polygons with sides all given but one, the maximum (in area) can be inscribed in a semicircle having the undetermined side for its diameter.*

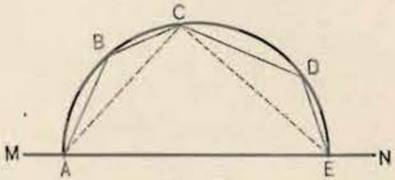


FIG. 159

Given the lengths of all sides of the polygon $ABCDE$, except EA; and given that $ABCDE$ is the maximum polygon (in area) that has the given sides and any other side EA.

To prove that $ABCDE$ can be inscribed in a semicircle whose diameter is the remaining side, EA.

Proof. From any vertex, as C, draw CA and CE.

The $\triangle ACE$ must be the maximum of all $\triangle$ having the given sides CA, CE; otherwise, by increasing or diminishing the $\angle ACE$, meanwhile keeping the lengths of CA, CE unchanged, as also the form of the figures, ABC, CDE, but allowing A and E to slide along MN while C is lowered or raised, we can increase the $\triangle ACE$, while the rest of the polygon remains unchanged in area. Hence, unless $\triangle ACE$ is the maximum, we can by such processes increase the area of $ABCDE$. To increase in this way the area of $ABCDE$ is, however, to deny the hypothesis that $ABCDE$ is the maximum polygon. Hence, the $\triangle ACE$ is the maximum that can be drawn, having the sides AC, CE.

Therefore, $\triangle ACE$ is a right triangle. § 227

Hence, also, C lies on the semicircumference of which AE is diameter. Why?

Likewise, every vertex can be shown to lie on the semicircumference whose diameter is AE; that is, the maximum polygon can be inscribed as stated in the theorem.

233. Theorem V. *Of all polygons with the same given sides, that which can be inscribed in a circle is the maximum.*

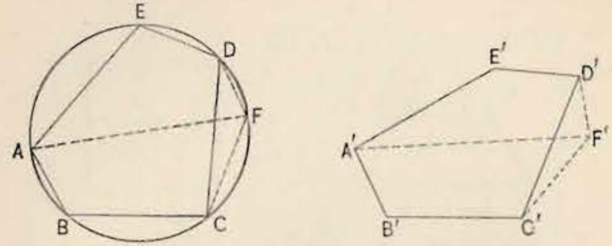


FIG. 160

Given a polygon $ABCDE$ inscribed in a circle and mutually equilateral to another given polygon, $A'B'C'D'E'$, which cannot be inscribed.

To prove $ABCDE > A'B'C'D'E'$.

Proof. From A draw a diameter AF and join F to the adjacent vertices, D and C. On $C'D'(= CD)$ construct $\triangle C'F'D' \cong \triangle CDF$ and draw $A'F'$.

Then, $AEDF > A'E'D'F'$ and $ABCF > A'B'C'F'$. § 232

Adding these two inequalities,

$$ABCFDE > A'B'C'F'D'E'.$$

Take away from the two figures the equal $\triangle$ CFD, $C'F'D'$, and we have, $ABCDE > A'B'C'D'E'$.

234. Corollary 1. *Of all isoperimetric polygons of a given number of sides, the maximum is regular.*

EXERCISE

1. If in the figure of § 232 we regard the sides AB, BC, etc., as stiff rods and assume that the rod AB is attached to the rod BC by means of a hinge at B, with a similar arrangement at each of the joints, we have what is known as a jointed frame. Considering all the different forms which this jointed frame $ABCDE$ can assume, what can be said of that one whose area is the maximum?

235. Theorem VI. *Of two isoperimetric regular polygons, the one having the greater number of sides has the greater area.*

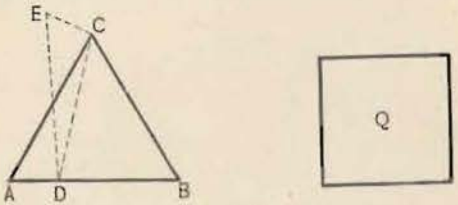


FIG. 161

Given a regular polygon of three sides (equilateral triangle) ABC and a regular polygon of four sides (square) Q, and let ABC and Q be isoperimetric.

To prove that $Q > ABC$.

Proof. Draw CD from C to any point in AB, and upon CD construct a triangle EDC which shall be congruent to ADC; that is, such that $ED = AC$ and $EC = AD$.

The figure $DBCE$ thus formed is an irregular polygon of four sides which by construction has the same perimeter as ABC, and hence the same as Q. Also, it has by construction the same area as ABC.

But $Q >$ the irregular polygon $DBCE$. § 231

Whence $Q > ABC$.

In like manner it can be shown that a regular polygon of five sides is greater than an isoperimetric square, and so on.

236. Corollary 1. *The circle is the maximum of all isoperimetric plane closed figures.*

EXERCISES

1. Give the proof that a regular polygon of five sides is greater than a corresponding isoperimetric square.

2. Show that a round can will hold more than a square can of the same perimeter, the two cans being of equal height.

237. Theorem VII. *Of all regular polygons of the same area, that which has the greatest number of sides has the minimum perimeter.*

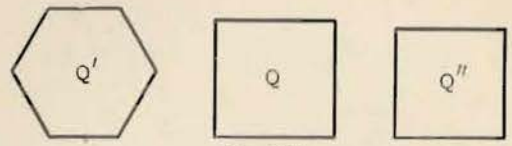


FIG. 162

Given the equivalent regular polygons Q and Q', of which Q' has the greater number of sides.

To prove that the perimeter of $Q' >$ the perimeter of Q.

Proof. Construct a regular polygon Q'' having the same perimeter as Q' and the same number of sides as Q.

Then $Q' > Q''$. Therefore $Q > Q''$. § 235

Whence the perimeter of $Q >$ the perimeter of Q''. § 182

But the perimeter of $Q' =$ the perimeter of Q''. Cons.

Therefore the perimeter of $Q >$ perimeter of Q'.

238. Corollary 1. *Of all plane closed figures that are equal in area, the circle has the minimum perimeter.*

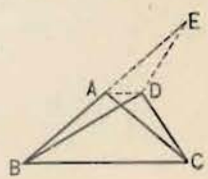
MISCELLANEOUS EXERCISES. MAXIMA AND MINIMA

1. From two given points on the circumference of a circle to draw two lines meeting on a tangent to the circle and making a maximum angle with each other.

2. To inscribe the maximum rectangle in a circle.

3. To inscribe the maximum rectangle in a quadrant.

4. Prove that of all triangles having the same base and area, the isosceles triangle has the minimum perimeter.



[HINT. Prolong AB to E, making $AE = AC$. Draw ED and AD. Prove that $\triangle ACD \cong \triangle AED$ so that $DC = DE$. Now use 3, § 28.]

SOLID GEOMETRY

CHAPTER VI

LINES AND PLANES IN SPACE

PART I. GENERAL PRINCIPLES

239. Definitions. **Solid Geometry**, or **Geometry of Three Dimensions**, treats of figures whose parts are not confined to a plane.

A **plane** is a surface such that if any two points in it are taken, the straight line passing through them lies wholly in the surface.

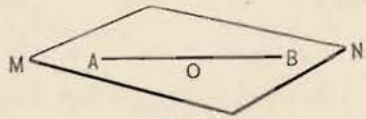


FIG. 163

Thus, in Fig. 163, if A and B are two points of a plane MN, the entire straight line AB lies in the plane MN. Any point O on AB lies in MN.

A plane is said to be **determined** by certain points and lines if that plane and no other plane contains those points and lines.

240. Corollary 1. It is evident from the definition of a plane that *if a line has two of its points in a plane, it lies wholly in that plane.*

241. Assumptions, or Postulates.

1. *A plane is unlimited in extent.*

2. *Through any straight line an unlimited number of planes may be passed.* See Fig. 164.

3. *If a plane is revolved about any straight line in it as an axis, it may be made to pass through any point in space.*

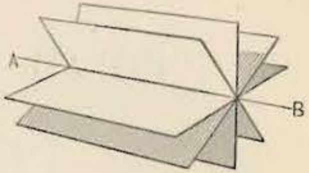


FIG. 164

4. *One and only one plane can be made to pass through three points not in the same straight line.*

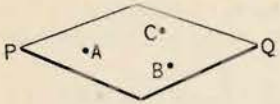


FIG. 165 (*a*)

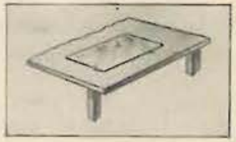


FIG. 165 (*b*)

Fig. 165 (*a*) represents a plane *PQ* through three points *A*, *B*, *C*. Fig. 165 (*b*) represents a plane piece of glass resting on the points of three tacks.

5. *Two planes cannot intersect each other in only a single point.*

242. Corollary 1. *A plane is determined by two intersecting lines.*

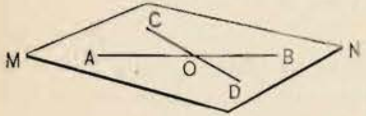


FIG. 166 (*a*)

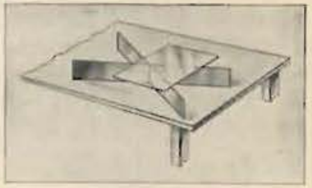


FIG. 166 (*b*)

[HINT. Consider the point where the lines intersect, and two other points, one on each line; then apply 4, § 241.]

243. Corollary 2. *A line and a point without the line determine a plane.*

[HINT. Use 4, § 241.]

FIG. 167 (*a*)

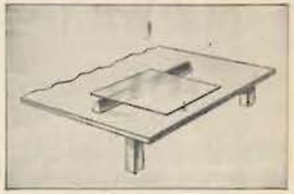


FIG. 167 (*b*)

244. Corollary 3. *Two parallel lines determine a plane.*

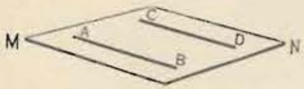


FIG. 168 (*a*)

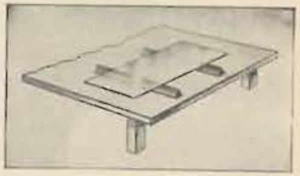


FIG. 168 (*b*)

[HINT. By the definition of parallel lines (§ 48), two such lines *must* lie in a plane. Show that this is the only one.]

EXERCISES

1. How many planes pass through a given straight line in space? How many pass through *two* given points?

2. In a carpenter's plane the knife-edge lies along a straight line. As soon as any rough surface has been sufficiently planed off, the whole length of the knife-edge keeps on the surface as the plane is moved along. Connect this fact with § 240.

3. Why are cameras, surveyors' transits, etc., mounted on three legs instead of four?

4. Prove that a straight line can intersect a plane in but one point unless it lies wholly in the plane. See § 240.

245. Theorem I. *The intersection of two planes is a straight line.*

Given the two intersecting planes MN and RS.

To prove that their intersection is a straight line.

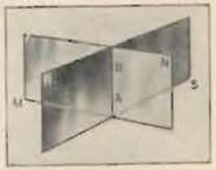


FIG. 169

Proof. Let A and B be any two points common to both planes. 5, § 241

Draw the straight line AB.

Then every point in AB lies in MN and also in RS. § 240

Therefore, AB is common to the two planes.

Moreover, no point not on AB can be common to both planes, for the two planes would then coincide. 4, § 241

Therefore, the intersection of the planes MN and RS is a straight line.

EXERCISES

1. What is the *locus* of all points common to two intersecting planes?

2. If a sheet of paper is folded, why is the crease straight?

3. In how many points (in general) will three planes intersect? What can be said of the intersection of four or more planes in space?

4. Can two pencils be held in such a position that a plane cannot be passed through them? State the *general* fact about a plane that is illustrated by your answer.

5. Can a plane be passed (in general) through four or more given points in space? Can a plane be passed (in general) through three lines all of which pass through a common point in space?

6. Can there be two straight lines that are not parallel and that do not meet? Find a pair of such lines in Fig. 169.

PART II. PERPENDICULARS AND PARALLELS

246. Line Perpendicular to a Plane. The point where a line intersects a plane is called the **foot** of the line on that plane.

A straight line is **perpendicular** to a plane when it is perpendicular to every straight line in the plane drawn through its

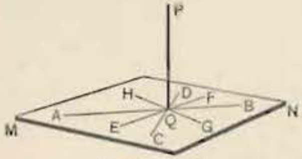


FIG. 170 (*a*)

FIG. 170 (*b*)

foot. The plane is then also said to be perpendicular to the line. Thus, in Fig. 170 (*a*), if PQ is perpendicular to the plane MN, it is then perpendicular to all the lines QA, QB, QC, etc.; and PQ is called the **distance** from P to MN; see Ex. 1 below.

247. Parallel Planes and Lines. A straight line is parallel to a plane if they never meet, however far produced. Two planes are parallel if they never meet, however far produced.

It is to be remembered (§ 48) that two lines are parallel only when they *lie in the same plane* and do not meet.

248. Corollary 1. *A plane that contains one of two parallel lines is parallel to the other line.*

EXERCISES

1. Show, by § 77, that the perpendicular from a point P to a plane MN (Fig. 170 *a*) is shorter than any other line that can be drawn from P to MN.

2. Show, by § 71, that if two oblique lines from a point P to a plane MN cut off equal distances from the foot of the perpendicular from P to MN, they are equal. See Ex. 1, p. 63.

249. Theorem II. *If a line is perpendicular to each of two lines at their point of intersection, it is perpendicular to their plane.*

Given FB perpendicular at B to each of two straight lines AB and BC of the plane MN.

To prove FB perpendicular to the plane MN.

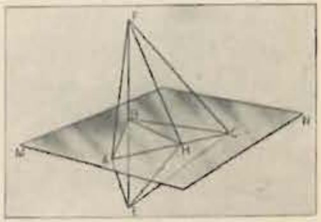


FIG. 171

Proof. Draw AC, and through B draw any line, as BH, meeting AC at H.

Prolong FB to E so that $BE = FB$. Join F and E to A, H, and C.

Then AB and BC are perpendicular bisectors of FE,		Const.
whence	$FA = AE$, $FC = CE$.	§ 100
Therefore	$\triangle AFC \cong \triangle AEC$,	§ 45
whence	$\angle HAF = \angle HAE$.	Why?
Also	$\triangle HAF \cong \triangle HAE$,	Why?
and	$HF = HE$.	Why?
Hence	$HB \perp FE$ or FB.	Why?
But HB was any line in MN drawn through B.		
Therefore	$FB \perp MN$.	§ 246

250. Corollary 1. *At a point in a plane only one perpendicular line can be erected.*

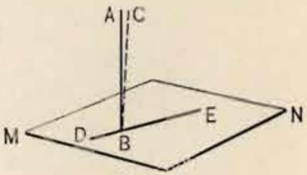


FIG. 172

[HINT. Suppose a second perpendicular line BC could be erected (Fig. 172). Pass a plane through AB and BC. This plane will intersect MN in a straight line, as DE. Then AB and BC are both perpendicular to DE at the same point B. But, since AB, BC, and DE all lie in the same plane, this is impossible, by 7, § 31.]

251. Corollary 2. *From a point without a plane, only one line can be drawn perpendicular to the plane.*

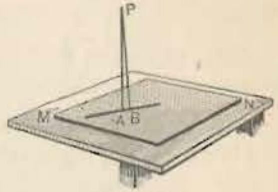


FIG. 173

[HINT. If two perpendiculars, as PB and PA, could be drawn from P to the plane MN, then $\triangle PBA$ would contain two right angles so that the sum of the angles of $\triangle PBA$ would be more than two right angles. But this is impossible. Why?]

252. Corollary 3. *Through a given point in a straight line, only one plane can be drawn perpendicular to the line.*

[HINT. Draw two different perpendiculars in space to the given line at the given point, and apply §§ 242, 249. If two such planes exist, their intersections with a plane through the given line violate 7, § 31.]

253. Corollary 4. *Through a given point without a straight line, only one plane can be drawn perpendicular to the line.*

[HINT. Prove by reduction to an absurdity. Show that the intersections of *two* such perpendicular planes with the *plane* determined by the given line and given point would violate § 58.]

254. Corollary 5. *All perpendicular lines that can be drawn to a straight line at a given point in it lie in a plane perpendicular to the line at the given point.*

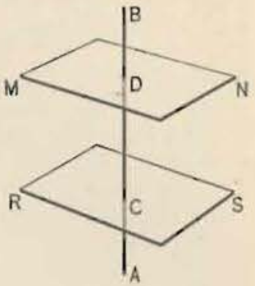
[HINT. Show that otherwise two perpendicular lines could be drawn to the given line in the same plane at the given point, thus violating 7, § 31.]

EXERCISES

1. Show how to determine a perpendicular to a plane by means of two carpenter's squares.

2. Tell how to test whether or not a flagpole is erect.

3. A spoke of a wheel is perpendicular to the axis on which it turns. Show by § 254 that it describes a plane in its rotation.



255. Theorem III. *Two planes perpendicular to the same line are parallel.*

[HINT. Show that if the two planes met, say in a point P, § 253 would be violated.]

FIG. 174

256. Theorem IV. *If a plane intersects two parallel planes, the lines of intersection are parallel.*

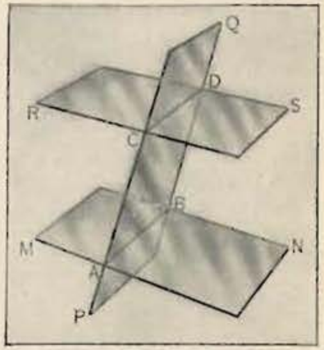


FIG. 175

Given the plane PQ intersecting the parallel planes MN and RS in AB and CD, respectively.

To prove $AB \parallel CD$.

[HINT. Prove, by reduction to an absurdity, that AB and CD cannot meet.]

257. Theorem V. *Two lines parallel to a third line (in space) are parallel to each other.* Compare § 50.

[HINT. Let BB' and CC' be two lines parallel to a third line AA' (Fig. 176). The plane determined by CC' and the point B on BB' is parallel to AA' (§ 248). Therefore (§ 48) the line of intersection of this plane with the plane of the parallels AA' and BB' is parallel to AA'. Hence show, by § 49, that this line of intersection coincides with BB', so that BB' and CC' lie in a plane. Finally, show that BB' and CC' cannot meet; for, if they did meet, say at a point D, the plane determined by D and AA' would contain (§ 244) both BB' and CC'.]

258. Theorem VI. *If two angles, not in the same plane, have their sides respectively parallel and extending in the same direction, they are equal and their planes are parallel.*

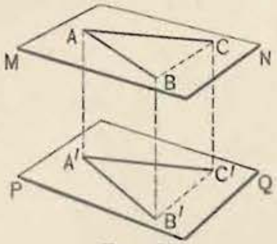


FIG. 176

Given the angles BAC and $B'A'C'$, lying in the planes MN and PQ, respectively, with $AB \parallel A'B'$, and $AC \parallel A'C'$.

To prove that $\angle A = \angle A'$, and that $MN \parallel PQ$.

Proof. Take $AB = A'B'$, and $AC = A'C'$.

Draw AA', BB', CC', CB, and $C'B'$.

Since AB is equal and parallel to $A'B'$,

it follows that $ABB'A'$ is a parallelogram; Why?

hence AA' is equal and parallel to BB'. Why?

Similarly, AA' is equal and parallel to CC'.

Hence BB' is equal and parallel to CC'. § 257

Then $BB'C'C$ is a parallelogram, and $CB = C'B'$. Why?

Therefore $\triangle ABC \cong \triangle A'B'C'$. Why?

Hence $\angle A = \angle A'$. Why?

Now $PQ \parallel AB$. Likewise $PQ \parallel AC$. § 248

Therefore, $PQ \parallel MN$ for, if not, the line of intersection of PQ and MN would meet either AB or AC (or both) extended; hence PQ would not be parallel to each of them.

NOTE. The similar theorem for angles that lie in the same plane was proved in § 67. As in § 67, the two angles are *supplementary* to each other if one pair of corresponding sides extend in *opposite* directions from the vertices.

259. Theorem VII. *A plane perpendicular to one of two parallel lines is perpendicular to the other also.*

Given the two parallel lines AB and CD, and a plane MN perpendicular to CD at C.

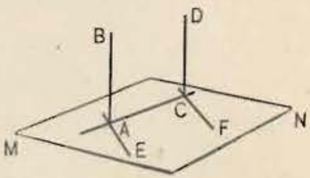


Fig. 177

To prove that MN is perpendicular to AB.

Proof. The parallel lines AB and CD determine a plane (§ 244) which intersects MN in some line AC. § 245

Now AC is perpendicular to CD; § 246

whence AC is perpendicular to AB. § 60

Draw any line AE in the plane MN through A.

Draw CF in MN parallel to AE through C.

Then CF is perpendicular to CD. § 246

Hence AE is perpendicular to AB. § 258

Therefore AB is perpendicular to MN. § 246

260. Corollary 1. *Two lines perpendicular to the same plane are parallel.*

[Hint. Let AB and CD (Fig. 177) be perpendicular to the plane MN. Imagine a parallel CD' to AB through C. Then CD' is perpendicular to MN, by § 259. Hence CD' coincides with CD, by § 250.]

EXERCISES

1. The legs of a table lie along parallel lines in space. What preceding theorem or corollary is illustrated here? Mention other similar illustrations.

2. How many lines can be drawn through a given point parallel to a given plane? If there is more than one such, what is the locus of them all?

3. Given a plane and two points without it. When will the line through the two points be parallel to the plane?

261. Theorem VIII. *If two straight lines are intersected by three parallel planes, the corresponding segments of these lines are proportional.*

Given the straight lines AB and CD cut by the parallel planes L, M, and N.

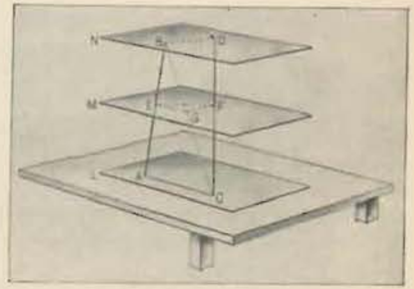


Fig. 178

To prove that

$AE/EB = CF/FD$.

Proof. Draw BC meeting the plane M in G. Draw EF, EG, FG, BD, and AC.

Then $GF \parallel BD$, and $EG \parallel AC$. § 256

Now $AE/EB = CG/GB$, and $CF/FD = CG/GB$. § 145

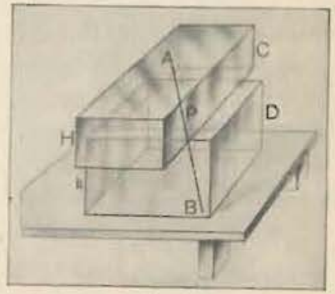
Therefore $AE/EB = CF/FD$. Why?

EXERCISES

1. Show that if parallel planes intercept equal segments on one line, they will intercept equal segments on any other line.

2. In Fig. 178, $AE = 5$, $EB = 4$, and $CF = 6$. What is the value of FD?

3. Two ordinary blocks C and D having the respective heights H and h are placed upon each other as shown in the figure. Show that any line AB drawn from the upper surface of C to the lower surface of D will be divided in the ratio $H : h$ by the point P where AB intersects the common surface of the two blocks.



262. Perpendicular Planes. Two planes MN and PQ are said to be **perpendicular** to each other when any line CD drawn in the one perpendicular to their intersection is perpendicular to the other plane.

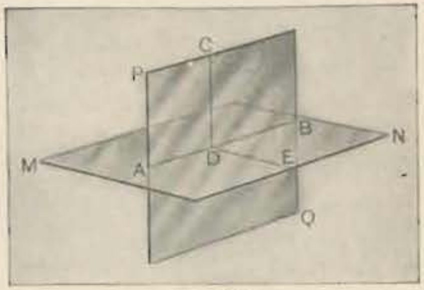


FIG. 179.

263. Theorem IX. *If a straight line is perpendicular to a plane, every plane containing this line is perpendicular to the given plane.*

Given the line CD perpendicular to plane MN; and given any plane PQ containing the line CD.

To prove that plane PQ is perpendicular to plane MN.

Proof. Let AB be the line of intersection of the two planes MN and PQ. Imagine any line $C'D'$ in the plane PQ perpendicular to the line AB.

Then	CD is parallel to $C'D'$.	§ 52
But	CD is perpendicular to MN;	Given
hence	$C'D'$ is perpendicular to MN.	§ 259

Since $C'D'$ is any line of the plane PQ perpendicular to AB, it follows that PQ is perpendicular to MN. § 262

264. Corollary 1. *The line perpendicular to a given plane at a given point lies in any plane through that point perpendicular to the given plane.*

265. Theorem X. *If each of two intersecting planes is perpendicular to a third plane, their line of intersection is perpendicular to the third plane.*

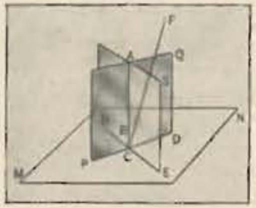


FIG. 180

Given the planes PQ and RS perpendicular to plane MN and intersecting each other in AB.

To prove that AB is perpendicular to MN.

Proof. Suppose that AB is not perpendicular to MN, but that some other line as CF through C, the point common to the three planes, is the perpendicular to MN.

Then CF lies in RS and in PQ.	§ 264
Hence CF coincides with AB.	§ 245

Therefore AB is perpendicular to MN at C.

EXERCISES

1. The blades of a side paddle wheel of a steamboat are all perpendicular to the side of the boat. Connect this fact with one of the preceding theorems. Do the same with the fact that the upright edge of any building is vertical.

2. How many planes can be drawn perpendicular to a given plane and passing through a given line in space?

[HINT. Select a point in the given line, draw the perpendicular line through that point to the given plane, and consider all the planes that can be passed through this perpendicular.]

PART III. DIHEDRAL ANGLES

266. Dihedral Angles. The figure formed by two intersecting portions of planes bounded by their line of intersection is called a **dihedral** angle. The planes forming the dihedral angle are its **faces** and the line of intersection is its **edge**.

A dihedral angle may be designated by the two letters on its edge, or by the two letters on its edge together with an additional letter on each face.

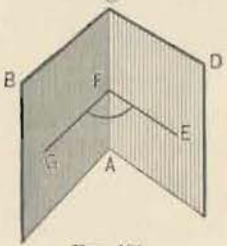


FIG. 181

Thus, in the figure, the planes AD and AB are the faces and AC is the edge of the dihedral angle $B\text{-}CA\text{-}D$.

The **plane angle of a dihedral angle** is an angle formed by lines in the two faces perpendicular to the edge at the same point. Thus, GFE is the plane angle of the dihedral angle $B\text{-}CA\text{-}D$.

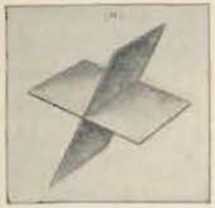
The magnitude of a dihedral angle does not depend upon the extent of its faces. If a plane be made to revolve from the position of one face about the edge as an axis to the position of the other face, it turns through the dihedral angle, and the greater the amount of turning, the greater the angle.

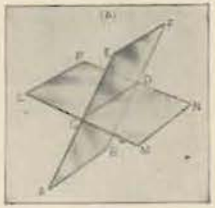
267. Measure of Dihedral Angles. The plane angle of a dihedral angle is taken as its **measure**, so that two dihedral angles are always in the same ratio as the magnitudes of their plane angles. In particular, two dihedral angles are **equal** when their plane angles are equal.

Dihedral angles are right, acute, obtuse, etc., according as their plane angles are right, acute, obtuse, etc. Similar definitions exist for complementary dihedral angles, supplementary dihedral angles, vertical dihedral angles, etc. The faces of a right dihedral angle are perpendicular to each other.

EXERCISES

1. Read the adjacent dihedral angles in the following figure. Read the vertical, the complementary, the supplementary dihedral angles.



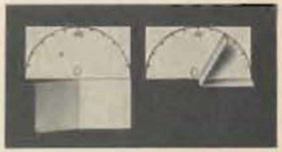


2. If two planes intersect each other, show that the opposite or vertical dihedral angles thus formed are equal.

[HINT. Use § 267.]

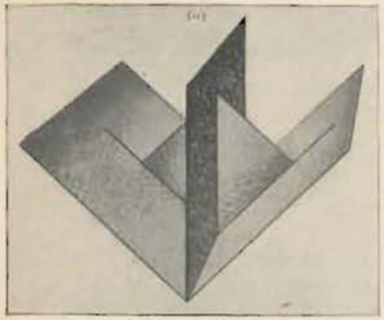
3. Show that the dihedral angle through which a door is opened is measured by the plane angle through which the bottom edge of the door moves.

4. Make an instrument for measuring dihedral angles by cutting and folding a piece of heavy paper or cardboard in the manner shown in the figure.



5. What is the number of degrees in one of the dihedral angles of a bay window, it being understood that the bay window consists of three equal upright plane sections, and that their bases form three sides of a regular octagon?

268. Theorem XI. *Every point in a plane that bisects a dihedral angle is equidistant from the faces of the angle.*



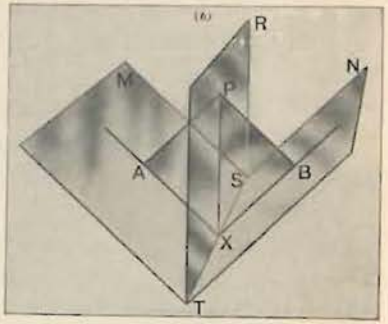


FIG. 182

Given the plane TR bisecting the dihedral angle formed by the planes TM and TN, so that the dihedral angles $M\text{-}TS\text{-}R$ and $N\text{-}TS\text{-}R$ are equal; and **given** PA and PB perpendicular to TM and TN, respectively, from any point P in TR.

To prove that $PA = PB$.

Proof. Pass a plane through PA and PB and let X be its point of intersection with ST; let AX, BX, and PX be the intersections of plane PAB with planes TM, TN, and TR.

Plane $PAB \perp$ planes TM and TN. § 263

Then plane $PAB \perp ST$, § 265

whence $ST \perp AX$, BX, and PX. Why?

The angles AXP and BXP are the plane angles of the dihedral angles $M\text{-}ST\text{-}R$ and $N\text{-}ST\text{-}R$. Why?

Since the dihedral angles are given equal, their plane angles are equal, that is, $\angle AXP = \angle BXP$;

whence rt. $\triangle AXP \cong$ rt. $\triangle BXP$, Why?

and therefore $PA = PB$. Why?

269. Corollary 1. *Any point not in the bisecting plane of a dihedral angle is unequally distant from the two faces.*

270. Corollary 2. *The plane bisecting a dihedral angle is the* ***locus*** *of all points equally distant from the faces of the angle.*

See NOTE, § 99.

EXERCISES

1. To what theorem in Plane Geometry does § 268 correspond?

2. From any point within a dihedral angle perpendiculars are drawn to the faces. Show that the angle formed by these perpendiculars is supplementary to the plane angle of the dihedral angle.

3. Prove that the two adjacent dihedral angles formed by one plane meeting another are supplementary.

[HINT. At some point on the edge of the dihedral, erect a plane perpendicular to its edge, and consider the plane angles formed.]

4. What is the locus of *all* points equidistant from two intersecting planes, each of indefinite extent?

5. What is the locus of all points in space equidistant from two given points?

6. What is the locus of all points in space equidistant from the circumference of a circle?

7. What is the locus in space of all points equidistant from two intersecting lines?

8. What is the locus of all points equally distant from two parallel lines?

9. Prove that of the dihedral angles formed by a plane intersecting parallel planes, the alternate and corresponding angles are equal, and the interior angles on the same side of the transversal plane are supplementary.

10. Prove that all plane angles of a dihedral angle are equal.

PART IV. POLYHEDRAL ANGLES

271. Polyhedral Angles. The figure formed by three or more straight line segments that end in a common point, together with the V-shaped portions of planes determined by pairs of adjacent lines, is called a **polyhedral angle.**

The point at which the lines all meet is called the **vertex** of the angle.

The lines in which the planes meet are its **edges**; and the V-shaped portions of the planes between these edges are its **faces.**

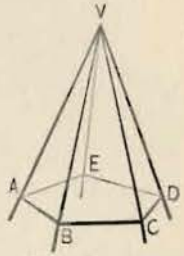


Fig. 183. Polyhedral Angle

The plane angles in the faces at the vertex are called the **face angles** of the polyhedral angle.

A polyhedral angle is *read* by naming the vertex and a point in each edge. Thus, in Fig. 183, the polyhedral angle is read $V\text{-}ABCDE$.

Two polyhedral angles are **congruent** if they can be placed so that their vertices coincide and their corresponding edges coincide.

A **trihedral angle** is a polyhedral angle that has three faces.

Thus, in Fig. 184, the three planes VAB, VBC, VAC, which meet at V form the trihedral angle $V\text{-}ABC$.

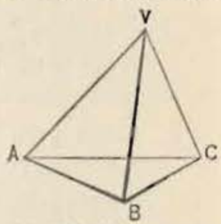


Fig. 184. Trihedral Angle

Two trihedral angles are **congruent** if the three face angles of the one are equal, respectively, to the three face angles of the other, and are arranged in the same order. This can be shown by methods similar to those of § 45. See also §§ 361, 374.

If the intersections of a plane with all the faces of a polyhedral angle is a convex polygon, the polyhedral angle is a *convex polyhedral angle.*

272. Theorem XII. *The sum of two face angles of a trihedral angle is greater than the third.*

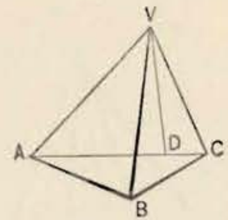


Fig. 185

Given the trihedral angle $V\text{-}ABC$.

To prove that $\angle AVB + \angle BVC > \angle AVC$.

Proof. If $\angle AVC$ is equal to or less than either of the other angles, we know the proposition is true without further proof.

If $\angle AVC$ is greater than either of the other angles, lay off any lengths VA and VC on the sides of $\angle AVC$, and draw AC. Then draw VD in the plane AVC, making $\angle AVD = \angle AVB$.

Lay off $VB = VD$, and draw AB and CB.

Then	$\triangle AVB \cong \triangle AVD$.	Why?
Therefore	$AB = AD$.	Why?
Now	$AB + BC > AD + DC$.	Why?
Whence, subtracting,	$BC > DC$.	Ax. 6
Therefore	$\angle BVC > \angle DVC$.	§ 79
By construction	$\angle AVB = \angle AVD$.	
Adding,	$\angle AVB + \angle BVC > \angle AVC$.	

EXERCISES

1. If in the trihedral angle $V\text{-}ABC$, $\angle AVB = 60°$, and $\angle BVC = 80°$, make a statement as to the number of degrees in $\angle AVC$.

2. Show that any face angle of a trihedral angle is greater than the difference of the other two.

273. Theorem XIII. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*

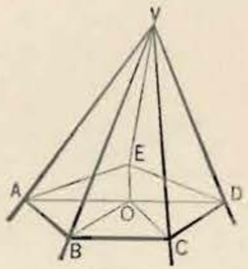


FIG. 186

Given the polyhedral angle $V\text{-}ABCDE$ with the edges cut by any plane in the points A, B, C, D, E.

To prove that the sum of the face angles of the polyhedral angle is less than four right angles.

Proof. Connect any point O in the polygon $ABCDE$ with the vertices A, B, C, D, E.

The number of triangles with the common vertex O is the same as the number having the vertex V.

Now $\angle VBA + \angle VBC > \angle ABO + \angle OBC$, § 272

and $\angle VAB + \angle VAE > \angle BAO + \angle OAE$, etc.

Therefore the sum of the base angles of the triangles having V for a common vertex is greater than the sum of the base angles of the triangles having O for vertex.

But the sum of all the angles of all the triangles whose vertex is V is equal to the sum of all the angles of all the triangles whose vertex is O. Why?

Therefore the sum of the angles about the vertex V is less than the sum of the angles about O, that is, less than four right angles.

MISCELLANEOUS EXERCISES ON CHAPTER VI

1. Lean one book in a slanting position against another book that lies flat on a table, and hold a stretched string parallel to the cover of the slanting book. Can the string have more than one position? Can the string be horizontal? Vertical?

2. Show that if a half-open book is placed on a table, resting on its bottom edges, the back edge of the book is perpendicular to the plane of the table, and the lines of printing are parallel to that plane.

3. Show that the dihedral angle between the pages of an open book is measured by the plane angle between opposite lines of type on the two pages.

4. What is the shape of the end of an ordinary plank after it has been sawed off in a slanting direction, assuming that the opposite faces of the original board are parallel planes?

5. Prove that the segments of two parallel lines included between parallel planes are equal.

[HINT. Pass a plane through the parallel lines and then prove that the given segments form the opposite sides of a parallelogram.

6. Prove that a plane perpendicular to the edge of a dihedral angle is perpendicular to both its faces.

[HINT. Use § 263.]

7. What is the locus of all the points equidistant from the three faces of a trihedral angle?

8. Show that the locus of any given point on a line segment of fixed length, whose ends touch two parallel planes, is a third plane parallel to the given planes.

9. Prove that if three lines are perpendicular to each other at a common point in space, each line is perpendicular to the plane of the other two, and that the planes of the lines (taken in pairs) are perpendicular to each other. Note how this is illustrated on a cube, or in a corner of a room, or in a corner of an ordinary box.

10. The trihedral angle formed when three planes meet each other, so that each is perpendicular to the other two is called a **trirectangular** trihedral angle.

Prove that the edges of a trirectangular trihedral angle are mutually perpendicular by pairs. See §§ 246, 265.

11. Prove that the space about a point is divided into eight congruent trirectangular trihedral angles by three planes mutually perpendicular by pairs at the point.

12. Prove that if a line is parallel to the intersection of two planes, it is parallel to each of the planes.

[Hint. Suppose that the line is *not* parallel to one of the planes and thus argue to an absurdity.]

13. Prove that if a line is parallel to each of two intersecting planes it is parallel to their intersection.

14. Prove that if a line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.

[Hint. Pass a plane through the given line perpendicular to the given plane and use § 267.]

15. Can a trihedral angle be formed by placing three equilateral triangles so that one vertex of each lies at the vertex of the trihedral angle? [Hint. Use § 273.]

16. Can a convex polyhedral angle be formed as in Ex. 15 by placing at its vertex one vertex of each of *four* equilateral triangles? Can this be done with *five* equilateral triangles? With six? With more than six?

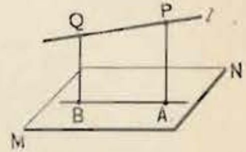
17. Can a convex polyhedral angle be formed by placing at its vertex one vertex of each of three squares? *Four* squares?

18. Can a convex polyhedral angle be formed by placing at its vertex one vertex of each of three regular pentagons? Four?

19. Show that just five different convex polyhedral angles can be formed as in Exs. 15–18 by placing at a single point one vertex of each of several similar regular polygons

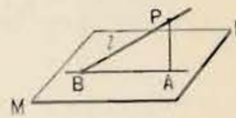
20. Show that the sum of the dihedral angles of a trihedral angle lies between two and six right angles.

21. Is there (in general) a point in space that is equidistant from four given points not all of which lie in the same plane? Give reason for your answer.



22. Given any line l and a plane MN, drop a perpendicular PA from any point P in l to MN. Prove that l and PA determine a plane perpendicular to MN. [This plane is called the **projecting plane** of l on MN. Its intersection AB with MN is called the **projection** of l on MN. Define similarly the projection of the *segment* PQ.]

23. Prove that the **projection** on a plane MN of the line segment joining two points P and Q (Ex. 22) is the line joining the feet A and B of the perpendiculars dropped to the plane from P and Q, respectively.



24. If a line l meets a plane MN at a point B, prove that the projection of l on MN is the line joining B to the foot A of a perpendicular let fall from any point P in l. [The angle ABP between the line l and its projection is called *the angle between the line and the plane*, or the *inclination of the line to the plane*.]

25. Prove that the sides of an isosceles triangle make equal angles with any plane containing its base.

***26.** Show that the length of the projection of any line segment PQ on any plane is the length of PQ times the cosine of the angle between the line and the plane.

CHAPTER VII

POLYHEDRONS CYLINDERS CONES

PART I. PRISMS

274. Polyhedrons. A **polyhedron** is a limited portion of space completely bounded by planes. The intersections of the bounding planes are called the **edges**; the intersections of the edges, the **vertices**; and the portions of the bounding planes bounded by the edges, the **faces**, of the polyhedron.

ICOSAHEDRON DODECAHEDRON OCTAHEDRON HEXAHEDRON TETRAHEDRON

FIG. 187

A polyhedron of four faces is called a **tetrahedron**; one of six faces (for example, a cube), a **hexahedron**; one of eight faces, an **octahedron**; one of twelve faces, a **dodecahedron**; one of twenty faces, an **icosahedron.**

A **diagonal** of a polyhedron is a straight line joining any two vertices not in the same face.

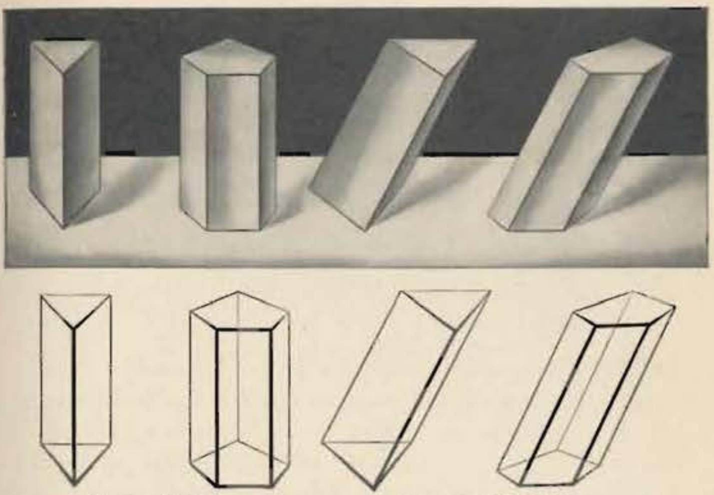
275. Prisms. A **prism** is a polyhedron, two of whose faces, called its ***bases***, are congruent polygons in parallel planes, and whose other faces, called ***lateral faces***, are parallelograms whose vertices all lie in the bases.

A **triangular** prism is one whose base is a triangle.

The sum of the areas of the lateral faces of any prism is called the **lateral area** of the prism.

The intersections of the lateral faces are the **lateral edges** of the prism.

The **altitude** of a prism is the perpendicular distance between its bases.



RIGHT PRISMS OBLIQUE PRISMS

FIG. 188

A **right prism** is one whose lateral edges are perpendicular to its bases.

An **oblique prism** is one whose lateral edges are oblique to its bases.

A **regular prism** is a right prism whose bases are regular polygons.

Any polygon made by a plane which cuts all the lateral edges of a prism, as the polygon $A'B'C'D'E'$ in Fig. 189, is called a **section** of the prism. A **right section** is one made by a plane perpendicular to all the lateral edges of the prism, as $ABCDE$.

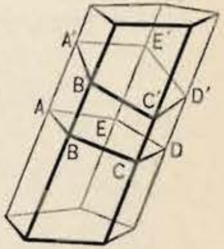


FIG. 189

276. Theorem I. *The sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.*

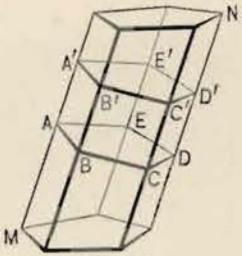


FIG. 190

Given $ABCDE$ and $A'B'C'D'E'$, sections of the prism MN, made by parallel planes.

To prove that $ABCDE \cong A'B'C'D'E'$.

Proof. The sides of the polygon $ABCDE$ are parallel to the sides of the polygon $A'B'C'D'E'$. § 256

Therefore the polygons are mutually equilateral. § 84

Also the polygons are mutually equiangular. § 258

Therefore polygon $ABCDE \cong$ polygon $A'B'C'D'E'$. § 33

EXERCISES

1. How many edges has a tetrahedron? A hexahedron?

2. How many diagonals has a hexahedron? A tetrahedron?

3. Prove that any two lateral edges of a prism are equal and parallel.

4. Prove that any lateral edge of a right prism is equal to the altitude.

5. Prove that all right sections of a prism are congruent.

6. Prove that a section of a prism parallel to the base is congruent to the base.

277. Theorem II. *The lateral area* **A** *of a prism is equal to the perimeter* p_r *of a right section multiplied by the lateral edge* e*; that is,* $A = p_r \times e$.

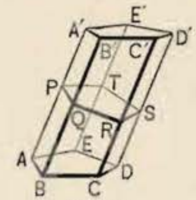


FIG. 191

Given the prism AD' with a right section $PQRST$; let p_r denote the perimeter of the right section, e the lateral edge, and A the lateral area.

To prove that $A = p_r \times e$.

Proof. The lateral area consists of a number of parallelograms, each of which has a line equal to AA' for its base. Why?

Each of these parallelograms has one of the sides of the right section $PQRST$ for an altitude. Why?

Therefore the areas of these parallelograms = the perimeter of $PQRST$ multiplied by AA'. Why?

That is $A = p_r \times e$.

278. Corollary 1. *The lateral area* **A** *of a right prism is equal to the perimeter of its base multiplied by a lateral edge*; or $A = p_b \times e$, where p_b denotes the perimeter of the base, and e denotes a lateral edge.

EXERCISES

1. Find the altitude of a regular prism, one side of whose triangular base is 5 in. and whose lateral area is 195 sq. in.

2. Show that the lateral area of a regular hexagonal right prism is $4\sqrt{3} \cdot ah$, where h is the altitude and a the distance from the center of the base to one of the sides.

279. Congruent Solids. Any two solids, in particular any two prisms, are said to be congruent when they can be made to coincide completely in all their parts.

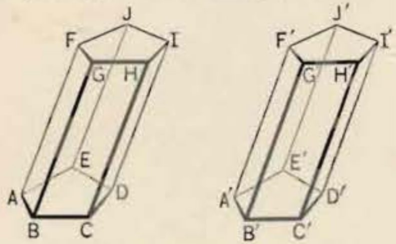


FIG. 192. CONGRUENT PRISMS

280. Theorem III. *Two prisms are congruent if three faces that include a trihedral angle of one are congruent respectively, and similarly placed, to three faces that include a trihedral angle of the other.*

Given the prisms AI and $A'I'$ with face $AJ \cong$ face $A'J'$, face $AG \cong$ face $A'G'$, and face $AD \cong$ face $A'D'$.

To prove that prism $AI \cong$ prism $A'I'$.

Proof. $\angle s$ EAF, FAB, and EAB are equal respectively to $\angle s$ $E'A'F'$, $F'A'B'$, and $E'A'B'$. Why?

Then trihedral $\angle A \cong$ trihedral $\angle A'$. § 271

Place the prism AI on the prism $A'I'$ so that the trihedral $\angle A$ coincides with its congruent trihedral $\angle A'$.

Then the face AJ will coincide with the congruent face $A'J'$; AG with the congruent face $A'G'$; and AD with $A'D'$; and points C and D will fall on C' and D'. § 33

Since the lateral edges of a prism are parallel and equal, CH coincides with $C'H'$, and DI with $D'I'$. §§ 257, 49

Therefore the upper bases coincide, and the prisms coincide throughout and are congruent.

281. Corollary 1. *Two right prisms having congruent bases and equal altitudes are congruent.*

282. Truncated Prisms. A **truncated prism** is a portion of a prism included between the base and a section oblique to the base.

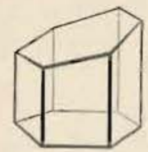


FIG. 193 (*a*)

FIG. 193 (*b*)

283. Corollary 2. *Two truncated prisms are congruent if three faces including a trihedral angle of the one are congruent respectively to three faces including a trihedral angle of the other.*

EXERCISES

1. A wooden beam 10 ft. long has a rectangular right cross section whose dimensions are 12 in. by 16 in. If the beam be sawed lengthwise along one of its diagonal planes, show that the resulting triangular prisms are congruent.

2. What will be the lateral area of one of the triangular prisms of Ex. 1? Its *total* area? *Ans.* 40 sq. ft.; $41\frac{1}{3}$ sq. ft.

3. A carpenter is to saw from a given square piece of timber a portion of which one end is to be perpendicular to the lateral edges, while three given lateral edges are to be 3 ft. 6 in., 3 ft. 4 in., and 3 ft. long, respectively. Show that these measurements are sufficient to enable him to saw off the desired portion.

4. Show that to make a right prism of any desired shape, it is sufficient to have a pattern of a right section of the desired prism, and the length of one lateral edge.

5. Show that to make a truncated prism of any desired shape, of which one end is a right section, it is sufficient to have a pattern of that end, and the lengths of three given consecutive lateral edges.

284. Theorem IV. *An oblique prism is equal in volume to a right prism whose base is a right section of the oblique prism and whose altitude is a lateral edge of the oblique prism.*

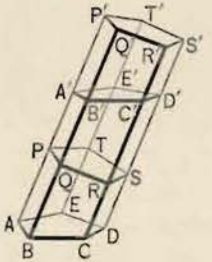


FIG. 194

Given the oblique prism AD'; and given a right prism PS' whose base PS is a right section of the prism AD', and whose altitude is equal to a lateral edge AA' of the prism AD'.

To prove that prism $AD' =$ prism PS'.

Proof. The lateral edges of the prism PS' equal the lateral edges of the prism AD'. Const.

Therefore $AP = A'P'$, $BQ = B'Q'$, etc. Why?

Moreover $PQ = P'Q'$, and the face angles at P, Q, P', Q' are right angles. Why?

Hence, by superposition,

Face $AQ \cong$ Face $A'Q'$.

Likewise, Face $BR \cong$ Face $B'R'$, etc.

Now, Section $PQRST \cong$ Section $P'Q'R'S'T'$. § 276

Whence, truncated prism $AR \cong$ truncated prism $A'R'$. § 283

Therefore, truncated prism $AR +$ truncated prism $PD' =$ truncated prism $A'R' +$ truncated prism PD'. Ax. 1

It follows that prism $AD' =$ prism PS'.

285. Equivalent Solids. Two solids that have the same volume are said to be **equivalent**, or **equal in volume**.

Thus we proved, in § 284, that any oblique prism is *equivalent* to a right prism whose base is a right section of the oblique prism and whose altitude is equal to the lateral edge of the oblique prism.

286. Parallelepipeds. A **parallelepiped** is a prism whose bases are parallelograms.

A **right parallelepiped** is a parallelepiped whose lateral edges are perpendicular to its bases.

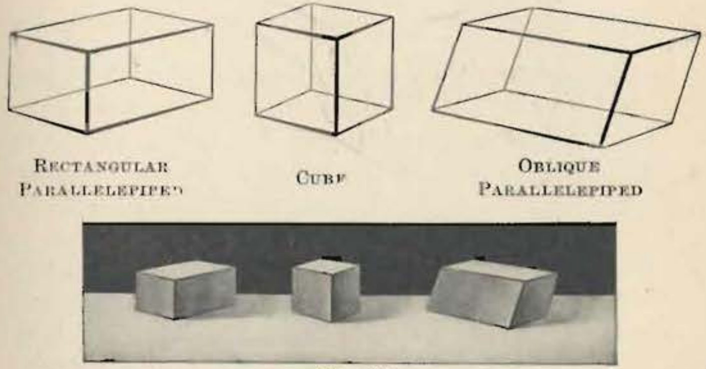


FIG. 195

A **rectangular parallelepiped** is a right parallelepiped whose bases are rectangles.

A **cube** is a parallelepiped whose six faces are squares.

An **oblique parallelepiped** is one whose lateral edges are oblique to its bases.

EXERCISES

1. Show that the lateral faces of a right parallelepiped are rectangles.

2. Find the sum of all the face angles of a parallelepiped.

3. Find the diagonal of a cube whose edge is 4 in.; 20 in.; a.

287. Theorem V. *The plane passed through two diagonally opposite edges of a parallelepiped divides the parallelepiped into two triangular prisms that are equal in volume.*

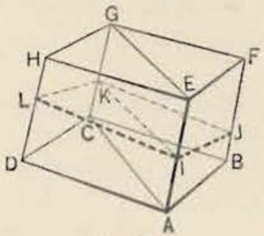


FIG. 196

Given the plane $ACGE$ passing through the opposite edges AE and CG of the parallelepiped AG.

To prove that the parallelepiped AG is divided into two equal triangular prisms, $ABC\text{–}F$ and $ADC\text{–}H$.

Proof. Let $IJKL$ be a right section of the parallelepiped.

From the definition of parallelepiped, the opposite faces, AF and DG, and AH and BG, are parallel and congruent. §§ 258, 88

Therefore $IJ \parallel LK$, and $IL \parallel JK$. § 256

Then $IJKL$ is a parallelogram. Why?

The intersection IK of the right section with the plane $ACGE$ is the diagonal of the $\square$ $IJKL$.

Therefore $\triangle IJK \cong \triangle IKL$. Why?

But the prism $ABC\text{–}F$ is equal in volume to the right prism whose base is IJK and altitude AE, and the prism $ACD\text{–}H$ is equal in volume to the right prism whose base is ILK and altitude AE. § 284

But since these right prisms are congruent, § 281

it follows that prism $ABC\text{–}F =$ prism $ADC\text{–}H$.

NOTE. If the faces $EFGH$ and $ABCD$ (Fig. 196) are perpendicular to the edges AE, BF, etc., it is easy to see that the diagonal plane $AECG$ divides the parallelepiped into two *congruent* triangular prisms. This happens for any rectangular parallelepiped.

EXERCISES

1. Prove that the diagonals of a rectangular parallelepiped are equal, and that the square of the diagonal is equal to the sum of the squares of the three edges that meet in any vertex.

[HINT. The diagonal is the hypotenuse of a right triangle whose sides are one of the edges and a diagonal of one face; the diagonal of the face is the hypotenuse of a right triangle whose sides are two of the edges.]

2. Find the length of the diagonal of a rectangular parallelepiped whose edges are 8, 10, 12.

3. Find the edge of a cube whose diagonal is 64 in.

4. Prove that the diagonals of a parallelepiped bisect each other.

[HINT. Consider each pair of diagonals separately, and apply § 87 to the diagonal plane in which they lie.]

5. Prove that if the right section $IJKL$ (Fig. 196) of a parallelepiped is a rectangle, the two diagonal planes $ACGE$ and $BDHF$ divide the parallelepiped into four triangular prisms that are equal in volume.

6. A tank in the form of a rectangular parallelepiped that holds 100 gal. is divided into four compartments by two vertical diagonal planes. What is the capacity of each compartment?

7. A cube each of whose edges is 1 ft. long is called a *unit* cube; its volume is one cubic foot. If six such cubes are placed side by side in two rows of three each, they form a rectangular parallelepiped 2 ft. wide, 1 ft. high, and 3 ft. long. What is the volume of this parallelepiped? What is the volume of each of the triangular prisms into which it is divided by a diagonal plane?

8. How many unit cubes are there in a cube each of whose edges is 5 units long?

9. How many unit cubes are there in a rectangular parallelepiped 3 units long, 4 units wide, and 2 units high? What is the volume of this parallelepiped?

288. Volume of a Rectangular Parallelepiped. The three edges of a rectangular parallelepiped which meet at a common point are called its **dimensions**.

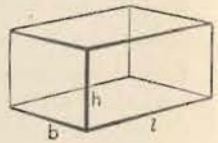


FIG. 197

In Chapter IV (§ 181), we assumed (without proof) the well-known principle that the area of a rectangle is equal to the product of its two dimensions. Similarly, we shall now assume that the volume of a rectangular parallelepiped is equal to the product of its *three* dimensions, that is, to the product of its length, breadth, and height; *i.e.*

For any **rectangular** *parallelepiped the volume* V *is*

$$V = l \times b \times h,$$

where l, b, h *denote the length, breadth, and height of the parallelepiped.*

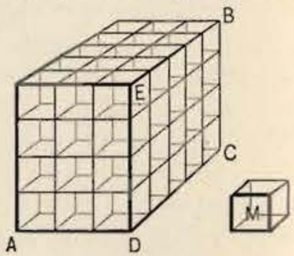


FIG. 198

The student is reminded that the meaning of the principle is that, if the three dimensions are each measured in terms of the *same* unit of length, then the volume in terms of the *corresponding unit cube* is the product of the three dimensions.

The following corollaries result at once from this principle:

289. Corollary 1. *Two rectangular parallelepipeds having congruent bases are to each other as their altitudes.*

[HINT. If l, b, and h represent the dimensions of the one parallelepiped, then l, b, and h' will represent the dimensions of the other. The corresponding volumes will therefore be to each other in the ratio $(lbh)/(lbh')$, that is, in the ratio h/h'.]

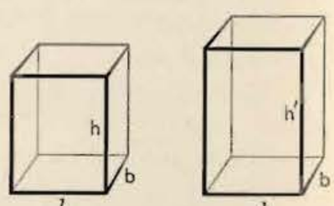


FIG. 199

290. Corollary 2. *Two rectangular parallelepipeds having equal altitudes are to each other as their bases.*

291. Corollary 3. *The volume of a cube is equal to the cube of its edge.*

292. Corollary 4. *The volume* V *of any* **rectangular** *parallelepiped is the product of the area of its base* B *and its altitude* h*; that is,* $V = B \times h$.

EXERCISES

1. Two rectangular parallelepipeds with equal altitudes have bases containing 10 sq. in. and 15 sq. in., respectively. The volume of the first is 56 cu. ft. Find the volume of the second. *Ans.* 84 cu. ft.

2. Compare the volume of the rectangular parallelepiped whose dimensions are 8 in., 10 in., 11 in. with the one whose dimensions are 1 ft., 1 ft., and 16 in.

3. In a lot 120 ft. long and 66 ft. wide a cellar is to be dug for a building. The cellar is to be 44 ft. long, 36 ft. wide, and 7 ft. deep. The earth removed is to be used to fill the surrounding yard. What depth of fill can be made?

4. A **standard** (U.S.) **gallon** contains 231 cu. in. How many gallons can be put in a tank of the form of a rectangular parallelepiped that is 2 ft. high, $1\frac{1}{2}$ ft. wide, and 3 ft. long?

5. How many gallons (see Ex. 4) are there in 1 cu. ft.?

6. Find the size of a cubical tank that will contain 50 gal.

7. Find the edge of a cube whose volume is 1728 cu. in.; of a cube whose volume is 1500 cu. in.

8. Find the diagonal of a cube whose volume is 521 cu. in.

9. If the volume of one cube is twice that of another, how do their edges compare? *Ans.* $\sqrt[3]{2}:1$.

10. Find the edge of a cube whose total surface is 60 sq. ft.

11. The edge of a cube is a. Find the area of a section made by a plane through two diagonally opposite edges.

293. Theorem VI. *The volume V of any parallelepiped is equal to the product of its base B and its altitude h; that is,* $V = B \times h$.

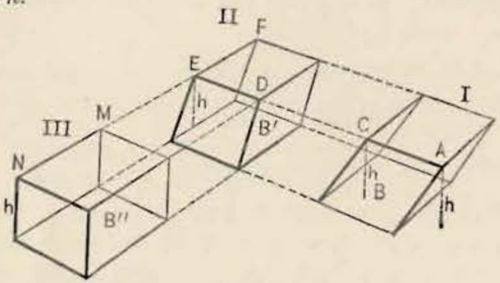


FIG. 200

Given the parallelepiped I with its volume denoted by V, its base by B, and its altitude by h.

To prove that $\qquad V = B \times h.$

Proof. Produce AC and all the other edges of I that are parallel to AC.

On the prolongation of AC take $DE = AC$, and through D and E pass planes perpendicular to AE, forming the right parallelepiped II whose base is B'.

Then $\qquad I = II.$ $\qquad$ § 284

Prolong FE and all the other edges of II that are parallel to FE.

On the prolongation of FE take $MN = FE$, and through M and N pass planes perpendicular to FN, forming the rectangular parallelepiped III whose base is B''.

Then	$II = III.$	Why?
Therefore	$I = III.$	Why?
Moreover	$B = B' = B''$;	Why?

and h, the altitude of I, is equal to the altitude of III. Why?

But the volume of III is $B'' \times h$, by § 288; hence the volume of I is $\qquad V = B'' \times h = B \times h.$

294. Theorem VII. *The volume V of any triangular prism is equal to the product of its base B and its altitude h; that is,* $V = B \times h$.

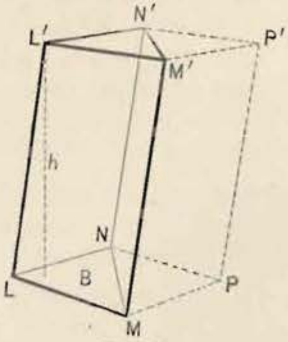


FIG. 201

Given the triangular prism $LMN\text{-}N'$ whose base B is the triangle LMN, and whose altitude is h.

To prove that the volume V of $LMN\text{-}N' = B \times h$.

Proof. Complete the parallelepiped $LMPN\text{-}P'$.

[The remainder of the proof is left to the student. Use § 293.]

295. Corollary 1. *The volume V of any prism is equal to the product of its base B and its altitude h; that is,* $V = B \times h$.

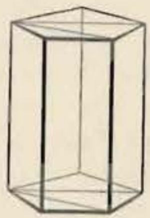


FIG. 202

[HINT. Any prism may be divided into triangular prisms by diagonal planes.]

296. Corollary 2. *Prisms having equivalent bases and equal altitudes are equal.*

EXERCISES

1. Describe one or more ways in which a parallelepiped may be distorted and yet have its volume remain unchanged.

2. The base of a parallelepiped is a rhombus one of whose diagonals is equal to its side. The altitude of the parallelepiped is a, and is also equal to a side of the base. Find the volume of the parallelepiped. *Ans.* $a^3\sqrt{3}/2$.

3. The altitude of a parallelepiped is 3 in., and a diagonal of a base divides the base into two equilateral triangles, each side of which is 6 in. Find the volume of the parallelepiped.

4. The volume of a rectangular parallelepiped is 2430 cu. in. and its edges are in the ratio of 3, 5, and 6. Find its edges.

5. The altitude of a prism is 6 in. and its base is a square each side of which is 3 in. Find its volume.

6. Show that two prisms with equal bases are to each other as their altitudes; and that those with equal altitudes are to each other as their bases.

7. A clay cube having a 2-in. edge is molded into the form of a triangular prism of height 3 in. What is the area of its base? Does it make a difference in the answer whether the prism is made right or oblique? Explain.

8. Assuming that iron weighs about 450 lb. per cu. ft., find the weight of a rod 3 ft. long, whose cross section is a rectangle $1\frac{1}{2}$ by 2 in.

9. With the data of Ex. 8, find the weight of an iron rod 2 ft. 6 in. long, whose cross section is a regular hexagon 1 in. on each side.

10. What must be the length of the side of an equilateral triangle in order that a triangular prism erected upon it and of height 1 ft. shall have a volume of 1 cu. ft.? Solve the same problem, when it is assumed that the base is a regular hexagon.

PART II. PYRAMIDS

297. Pyramids. A **pyramid** is a polyhedron bounded by a polygon, called its **base**, and several triangles that have a common vertex.

The triangles are called the **lateral faces**, the common vertex is called the **vertex** of the pyramid, and the perpendicular distance from the vertex to the base is called the **altitude**.

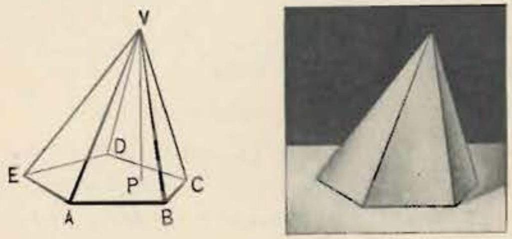


FIG. 203. PYRAMID

A pyramid is *triangular, quadrangular*, etc., according as its base is a triangle, a quadrilateral, etc.

A **regular pyramid** is one whose base is a regular polygon and whose vertex lies in the perpendicular erected at the center of the base.

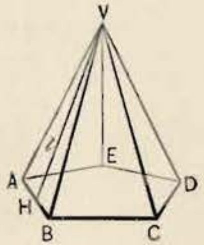


FIG. 204. REGULAR PYRAMID

The **slant height** of a *regular* pyramid is the altitude of any one of its triangular faces (VH in Fig. 204).

A **truncated pyramid** is the portion of a pyramid included between the base and any section made by a plane cutting all the lateral edges.

A **frustum of a pyramid** is a portion included between the base and a section made by a plane parallel to the base.

FIG. 205. FRUSTUM OF A PYRAMID

The *altitude of a frustum* is the length of the perpendicular between the planes of its bases.

The *lateral faces* of a frustum of a *regular* pyramid are congruent trapezoids.

The *slant height* of the frustum of a *regular* pyramid is the altitude of one of the trapezoids forming its faces.

EXERCISES

1. Of which type are the celebrated Egyptian pyramids?

2. Prove the equality of the lateral edges of a regular pyramid. Of those of a frustum of a regular pyramid.

3. Prove that the faces of any frustum of a pyramid are trapezoids.

4. Prove the statement made in § 297 that the faces of a frustum of a *regular* pyramid are *congruent* trapezoids.

5. Prove that the lateral faces of a regular pyramid are congruent isosceles triangles; hence show that the slant height, measured on any face, is the same as that measured on any other face.

6. Prove that any triangular pyramid is a tetrahedron. State and prove the converse.

298. Theorem VIII. ***The lateral area A of a regular pyramid is equal to one half the product of the perimeter of its base p, and its slant height l; that is, $A = p \times l/2$.***

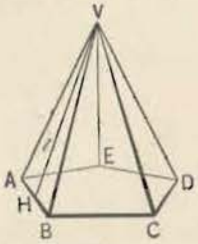


FIG. 206

Given the regular pyramid $V\text{-}ABCDE$ with the slant height l, the lateral area A, and the perimeter of the base p.

To prove that $A = p \times l/2$.

Proof. $A =$ the sum of the areas of the triangles VAB, VBC, etc.

Hence $A = [AB + BC + \cdots] \times l/2 = p \times l/2$. Why?

299. Corollary 1. *The lateral area of the frustum of a regular pyramid is equal to one half the product of the sum of the perimeters of the bases and its slant height.* [See § 191.]

EXERCISES

1. The slant height of a regular hexagonal pyramid is 10 ft. Each side of its base is 8 ft. What is its lateral area? Also, what is its total area? *Ans.* 240 sq. ft.; 406.27 sq. ft.

2. The altitude of a regular quadrangular pyramid is 4 in. One side of its base is 6 in. What is its lateral area? What is its total area? *Ans.* 60 sq. in.; 96 sq. in.

3. Find the lateral area of the frustum formed by a plane bisecting the altitude of the pyramid mentioned in Ex. 2 Find its total area.

300. Theorem IX. *If a pyramid is cut by a plane parallel to the base,*

(a) The altitude and the lateral edges are divided proportionally;

(b) The section is a polygon similar to the base.

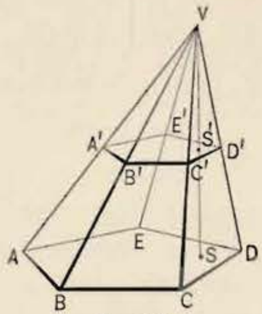


Fig. 207

Given the pyramid $V\text{-}ABCDE$ cut by a plane $A'D'$ parallel to the base AD.

To prove (a) that

$$VS/VS' = VA/VA' = VB/VB' = \cdots, \text{ etc.}$$

(b) that the section $A'B'C'D'E'$ is similar to the base.

Proof of (a)

Pass a plane through V parallel to the base and apply § 261.

Proof of (b)

$\triangle VAB \sim \triangle VA'B'$; $\triangle VBC \sim \triangle VB'C'$, etc. Why?

Therefore

$AB/A'B' = VB/VB'$; $VB/VB' = BC/B'C'$, etc.; Why?

and hence $AB/A'B' = BC/B'C' = CD/C'D' = \cdots$ Ax. 9

Thus, the polygons $ABCDE$ and $A'B'C'D'E'$ have their corresponding sides proportional.

Moreover, the same polygons are mutually equiangular. § 258

Hence $ABCDE \sim A'B'C'D'E'$. § 165

301. Corollary 1. *Parallel sections of a pyramid are to each other as the squares of their distances from the vertex.*

Proof: In Fig. 207, $ABCDE/A'B'C'D'E' = \overline{AB}^2/\overline{A'B'}^2$. §§ 195, 300

But $AB/A'B' = VB/VB'$, Why?

and also $VS/VS' = VB/VB'$; (a), § 300

hence $AB/A'B' = VS/VS'$. Ax. 9

Whence, squaring, $\overline{AB}^2/\overline{A'B'}^2 = \overline{VS}^2/\overline{VS'}^2$.

Hence $ABCDE/A'B'C'D'E' = \overline{VS}^2/\overline{VS'}^2$.

302. Corollary 2. *If two pyramids that have equivalent bases and equal altitudes are cut by planes parallel to their bases and at equal distances from their vertices, the sections are equivalent.*

[Hint. Represent by R and R' the areas of the two sections, and by B and B' the areas of the bases. Let h be the common altitude of the pyramids, and k the distance from the vertex of either pyramid to the section made in it. Then $R/B = k^2/h^2$ and $R'/B' = k^2/h^2$ (§ 301); hence $R/B = R'/B'$. But $B = B'$ by hypothesis; hence $R = R'$.]

EXERCISES

1. Compare the areas of two sections of a pyramid whose perpendicular distances from the vertex are 3 in. and 4 in. respectively. Does it make any difference in your answer whether the pyramid is of one shape or another? *Ans.* 9 : 16.

2. The altitude of a pyramid with a square base is 16 in., the area of a section parallel to the base and 10 in. from the vertex is $56\frac{1}{4}$ sq. in. Find the area of the base.

3. The bases of the frustum of a regular pyramid are equilateral triangles whose sides are 10 in. and 18 in. respectively; the altitude of the frustum is 8 in. Find the altitude of the pyramid of which the given figure is a frustum. *Ans.* 18 in.

4. The altitude of a pyramid is H. At what distance from the vertex must a plane be passed parallel to the base so that the section formed shall be (1) one half as large as the base? (2) one third? (3) one ninth?

303. Theorem X. *Two triangular pyramids having equivalent bases and equal altitudes are equivalent.*

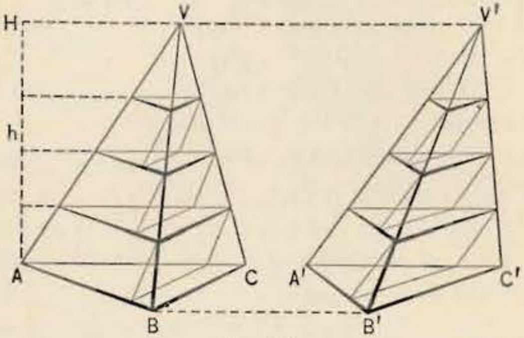


FIG. 208

Given the pyramids $V\text{-}ABC$ and $V'\text{-}A'B'C'$ having equivalent bases ABC, $A'B'C'$, and a common altitude AH.

To prove $V\text{-}ABC = V'\text{-}A'B'C'$.

Proof. The proof of the theorem consists in showing that the pyramids $V\text{-}ABC$ and $V'\text{-}A'B'C'$ cannot differ in volume by as much as any given amount, however small. This means that the two volumes are actually equal, for if they were unequal, they would differ by as much as some *fixed* amount, — in fact, that is what unequal means.

We proceed, then, to show that $V\text{-}ABC$ and $V'\text{-}A'B'C'$ cannot differ by as much as any given amount, however small.

Divide the altitude AH into a number of equal parts, and through the points of division pass planes parallel to the plane of the bases.

The corresponding sections made by any one of these planes in the two pyramids are equivalent. § 302

Now inscribe in each pyramid a series of prisms having the triangular sections as upper bases and the distance between the sections as their common altitude.

Each pair of corresponding prisms in the two pyramids are then equivalent. § 296

Therefore, the sum of the prisms inscribed in $V\text{-}ABC$ is equivalent to the sum of the prisms inscribed in $V'\text{-}A'B'C'$.

If the number of divisions into which AH is divided is taken sufficiently large, the sum of the prisms in $V\text{-}ABC$ may be made to differ from the volume of $V\text{-}ABC$ by less than any given amount. Likewise, by taking the number of divisions in AH sufficiently large, the sum of the prisms in $V'\text{-}A'B'C'$ may be made to differ from the volume of $V'\text{-}A'B'C'$ by less than the same given amount, however small it has been taken.

Since, by taking the number of divisions in AH sufficiently large, the volumes of $V\text{-}ABC$ and $V'\text{-}A'B'C'$ differ by less than any given amount from these *equal* sums, the pyramids must differ from each other by less than the same given amount, and this is what we were to show. Compare § 211.

304. Corollary 1. *Any two pyramids having equivalent bases and equal altitudes are equivalent.*

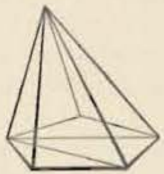


FIG. 209

[HINT. Divide each pyramid into triangular pyramids.]

EXERCISES

1. What is the locus of the vertices of all pyramids whose bases and volumes are the same?

2. Prove that if the base of a pyramid is a parallelogram, the plane determined by its vertex and either diagonal of its base divides it into two equivalent triangular pyramids.

305. **Theorem XI**. *The volume V of a triangular pyramid is equal to one third the product of its base B, and its altitude h; that is, $V = B \times h/3$.*

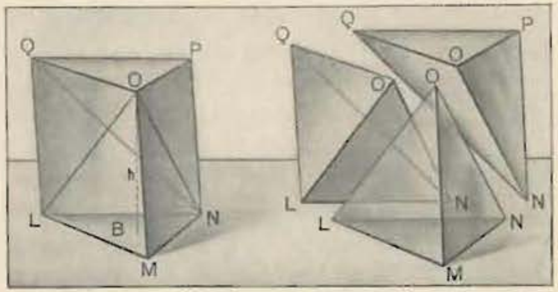


Fig. 210

Given the triangular pyramid $O\text{-}LMN$.

To prove that the volume V of $O\text{-}LMN$ equals $\frac{1}{3}$ its base B times its altitude h; that is, $V = B \times h/3$.

Proof. Construct a triangular prism MP having LMN for its base, and its lateral edges equal and parallel to the edge OM.

The prism MP is made up of the triangular pyramid $O\text{-}LMN$ and the quadrangular pyramid $O\text{-}LNPQ$.

Pass a plane through OQ and ON dividing the quadrangular pyramid into two triangular pyramids, $O\text{-}LNQ$ and $O\text{-}NQP$.

Pyramid $O\text{-}LNQ$ = pyramid $O\text{-}NQP$. § 303

Pyramid $O\text{-}NQP$ may be read $N\text{-}QOP$.

Pyramid $N\text{-}QOP$ = pyramid $O\text{-}LMN$. § 303

Therefore, the three triangular pyramids are equal and $O\text{-}LMN$ is one third the prism.

But the volume of the prism is equal to the product of its base and its altitude. § 294

Therefore, pyramid $O\text{-}LMN = \frac{1}{3}$ the product of its base and its altitude.

306. Corollary 1. *The volume V of any pyramid is equal to one third the product of its base B and its altitude h; that is, $V = Bh/3$.*

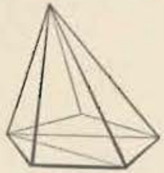


Fig. 211

[Hint. Divide the pyramid into triangular pyramids and apply § 305.]

307. Theorem XII. *If a frustum of a pyramid has bases B and B' and altitude h, and is cut from a pyramid P whose base is B and whose altitude is H, the volume V of the frustum is given by the formula:*

$$V = \frac{BH}{3} - \frac{B'(H-h)}{3}.$$

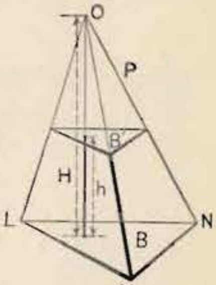


Fig. 212

Given the pyramid P with base B and altitude H, and a frustum of it with bases B and B' and altitude h.

To prove that the volume V of the frustum is

$$V = \frac{BH}{3} - \frac{B'(H-h)}{3}.$$

Proof. The frustum is the difference between the two pyramids P and P', where P' has the base B' and the same vertex as P.

The volume of P is $BH/3$. Why?

Since the altitude of P' is $H - h$, its volume is

$$\frac{B'(H-h)}{3}. \qquad \text{Why?}$$

Hence $$V = P - P' = \frac{BH}{3} - \frac{B'(H-h)}{3}.$$

308. Corollary 1. *The volume V of a frustum of a pyramid of bases B and B' and altitude h is given by the formula:*

$$V = (B + B' + \sqrt{BB'})h/3.$$

Outline of Proof. By § 301,

$$B'/B = (H-h)^2/H^2,$$

using the notation of § 307.

Hence $\sqrt{B'}/\sqrt{B} = (H-h)/H = 1 - h/H,$

whence $$H = h\sqrt{B}/(\sqrt{B}-\sqrt{B'}).$$

But, by § 307,

$$V = BH/3 - B'(H-h)/3$$
$$= (B-B')H/3 + B'h/3.$$

Substituting the value of H just found, we have

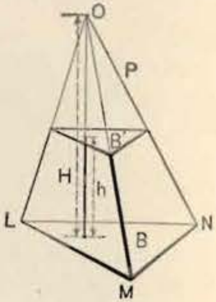


Fig. 213

$$V = \left[\frac{(B-B')\sqrt{B}}{\sqrt{B}-\sqrt{B'}} + B'\right]h/3 = [B + \sqrt{BB'} + B']h/3.$$

EXERCISES

1. Find the altitude of a triangular pyramid whose volume is 50 cu. in. and whose base is 12 sq. in. *Ans.* $12\frac{1}{2}$ in.

2. If a prism and a pyramid have a common base and altitude, what is the ratio of their volumes?

3. If the base of a pyramid is a square and its altitude is 3 ft., how long must each side of the square be in order that the volume may be 16 cu. ft.?

4. Show that the volume of the tetrahedron, all of whose edges are equal to a, is $\sqrt{2}\,a^3/12$.

[Hint. See Th. XXXIII, § 102.]

5. Find the volume of a frustum of the pyramid of Ex. 1 cut off by a plane 6 in. from the base.

6. Find the volume of each of the parts into which the pyramid of Ex. 3 is cut by two planes parallel to its base which trisect the altitude.

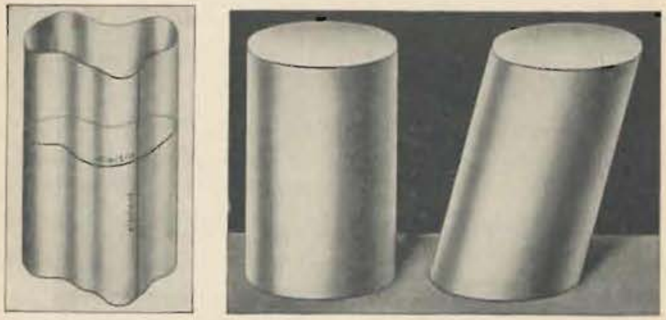
PART III. CYLINDERS AND CONES

309. Cylinders. A **cylindrical surface** is a curved surface generated by a moving straight line, called the **generatrix**, which moves always parallel to itself and constantly passes through a fixed curve called the **directrix**. The generatrix in any one position is called an **element** of the surface. One element and only one can be drawn through a given point on the cylindrical surface.

A **cylinder** is a solid bounded by a cylindrical surface and two parallel planes. The two plane surfaces are called the **bases**, and the cylindrical surface is called the **lateral surface**.

The **altitude** of a cylinder is the length of the perpendicular between the bases.

A **right section** of a cylinder is a section made by a plane perpendicular to all its elements.



Cylindrical Surface Right Cylinder Oblique Cylinder

Fig. 214

A **circular cylinder** is one whose bases are circles.

A **right cylinder** is one whose elements are all perpendicular to its bases; otherwise, the cylinder is said to be *oblique*.

A **right circular cylinder** is a right cylinder whose base is a circle. Such a cylinder can be generated by the revolution of a rectangle about one of its sides as an axis; for this reason a right circular cylinder is sometimes called a *cylinder of revolution*.

310. Postulate. A prism is **inscribed** in a cylinder when its lateral edges are elements of the cylinder and its bases are inscribed in the bases of the cylinder.

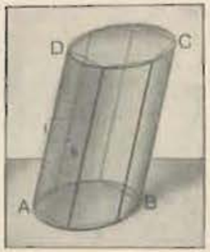


FIG. 215

In order to study the properties of the cylinder the following postulate is needed:

A cylinder and a prism inscribed within it may be made to differ by as little as we please, both in lateral area and in volume, by making the number of sides of the base of the prism sufficiently great, while the length of each of those sides becomes sufficiently small.

The length of an edge of the inscribed prism is equal to the length of an element of the cylinder (see Ex. 5, p. 235); and, by increasing the number of sides of the inscribed prism, the base of the prism approaches, both in area and in the length of its perimeter, as nearly as we please to the base of the cylinder. This latter fact we assume, as in § 210.

We shall now proceed to show that the theorems already proved for prisms can be extended to cylinders by the use of the preceding postulate.

311. Theorem XIII. *The lateral area* **A** *of any cylinder is equal to the product of an element* l *and the perimeter* p *of a right section; that is,* $A = l \times p$.

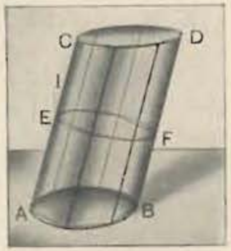


FIG. 216

Outline of Proof. In order to prove the theorem, inscribe in the cylinder any prism whose base is a polygon of n sides. Then, for this *prism*, by § 277:

(1) $$A' = l \times p',$$

where A' and p' are, respectively, the lateral area and the perimeter of the right section of the prism; and where l is the length of an edge of the prism, which is equal to an element of the cylinder. As the number n of sides increases so that the length of each side approaches zero,

A' comes to differ from A by as little as we please; § 310.

$l \times p'$ comes to differ from $l \times p$ by as little as we please. § 310.

Hence, by (1), A comes to differ from $l \times p$ by as little as we please.

It follows, as in § 303, that $A = l \times p$.

312. Corollary 1. *The lateral area of a right circular cylinder is equal to* $2\pi rh$, *where* r *is the radius of the circular base and* h *is the altitude of the cylinder.* See § 214.

313. Corollary 2. *The lateral area of any cylinder whose right section is a circle is equal to* $2\pi rl$, *where* r *is the radius of the right section, and* l *is the length of an element.*

314. Theorem XIV. *The volume V of any cylinder is equal to the product of its base B and its altitude h; that is, $V = B \times h$.*

[The proof is left to the student. Inscribe a prism of n sides in the cylinder and use § 295 and § 310. Follow the steps suggested by § 311.]

315. Corollary 1. *The volume of a circular cylinder is equal to $\pi r^2 h$, where r is the radius of the base and h is the altitude of the cylinder.* See § 216.

EXERCISES

[In these exercises, use the approximate value $\pi = 22/7$.]

1. In a steam engine 65 flues, or cylindrical pipes, each 2 in. in outside diameter and 12 ft. long, convey the heat from the fire-box through to the water. How much heating surface is presented to the water? *Ans.* $408\frac{4}{7}$ sq. ft.

2. Neglecting the lap, how much tin is required to make a stovepipe 10 ft. long and 8 in. in diameter?

3. A right circular cylinder has the radius of its base equal to 3 in. How great must its altitude be in order that it shall have a lateral area of 30 sq. in.?

4. Find the total area, including the ends, of a covered tin can whose diameter is 4 in. and whose height is 6 in.
Ans. $100\frac{4}{7}$ sq. in.

5. Derive a general formula for the total area (including the bases) of a right circular cylinder whose height is h and the radius of whose base is r.

6. What fraction of the metal in a tin can 5 in. wide and 5 in. high is used to make the top and bottom? What to make the circular sides?

7. If the diameter of a well is 7 ft. and the water is 10 ft. deep, how many gallons of water are there in it, reckoning $7\frac{1}{2}$ gal. to the cubic foot? *Ans.* 2887.5 gals.

8. When a body is placed in a cylindrical tumbler of water 3 in. in diameter, the water level rises 1 in. What is the volume of the body? Note that a method for finding the volume of a body of *any* shape is here illustrated.

9. Show that the volume V and the lateral area A of a right circular cylinder are connected by the relation $V = A \times r/2$.

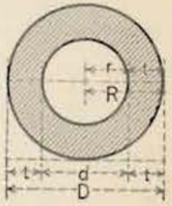
10. One gallon is 231 cubic inches. At what heights on a cylindrical measuring can whose base is 6 in. in diameter will the marks for 1 gallon, 1 quart, 2 quarts, 3 quarts, be made?

11. Find the total area of the gallon measuring can of Ex. 10.

12. Having given the total surface T of a right circular cylinder, in which the height is equal to the diameter of the base, find the volume V.

[HINT. — Call the height h.]

13. Find the volume of metal per foot of length in a pipe whose outer diameter is $3\frac{1}{2}$ in., and whose inner diameter is 3 in. Hence find the weight per foot of length if the pipe is iron, it being given that iron weighs (about) 450 lb. per cubic foot.



14. If the outer and inner diameters of a tube are D and d, respectively, show that the volume in a length l is $\pi l(D^2 - d^2)/4$. If the thickness of the tube is denoted by t, show that $t = (D - d)/2$, and hence that the volume in a length l is

$$\pi lt(D + d)/2.$$

15. Show that the volume of the tube of Ex. 14 can also be represented by the formula $\pi lt(d + t)$; or by the formula $\pi lt(D - t)$, in the notation of Ex. 14.

16. What is the volume of the largest beam of square cross section that can be cut from a circular log 16 in. in diameter and 10 ft. long? What fraction of the log is wasted?

316. Cones. A **conical surface** is a surface generated by a moving straight line AB, called the **generatrix**, which passes always through a fixed point V, called the **vertex**, and constantly

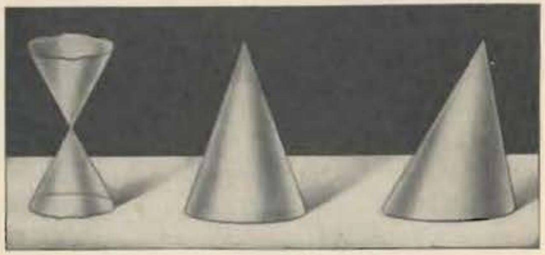


FIG. 217

intersects a fixed curve, called the **directrix**. A conical surface thus consists of two parts, which are called the upper and lower **nappes**. The generatrix in any one position is called an **element** of the surface.

A **cone** is a solid bounded by a conical surface and a plane which cuts all its elements. The plane is then called the **base** of the cone; and the conical surface is called the **lateral surface** of the cone. The **altitude** of a cone is the perpendicular distance from its vertex to its base.

FIG. 218

A **circular** cone is one whose base is a circle. The **axis** of a circular cone is the line joining the vertex to the center of the base.

A **right circular cone** is a circular cone whose axis is perpendicular to the base. Such a cone is also called a **cone of revolution**, since it may be generated by revolving a right triangle about one of its sides as an axis.

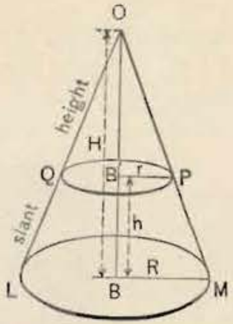


FIG. 219. RIGHT CIRCULAR CONE AND SECTION PARALLEL TO BASE

The **slant height** of a cone of revolution is the length of one of its elements.

A **frustum** of a cone is the portion of a cone included between the base and a section parallel to the base and cutting all the elements.

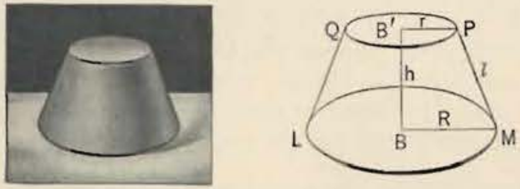


FIG. 220. FRUSTUM OF A CONE

The **lateral surface** of a frustum of a cone is the portion of the lateral surface of the cone included between the bases of the frustum.

The **slant height** of a frustum of a cone of revolution is the portion of any element of the cone included between the bases.

317. Postulate. A pyramid is **inscribed** in a cone when its lateral edges are elements of the cone and its base is inscribed in the base of the cone.

The following postulate, corresponding to that of § 310, is needed in the study of the cone:

A cone and pyramid inscribed within it may be made to differ by as little as we please, both in lateral area and in volume, by making the number of sides of the pyramid sufficiently great, while the length of each side of the base becomes sufficiently small.

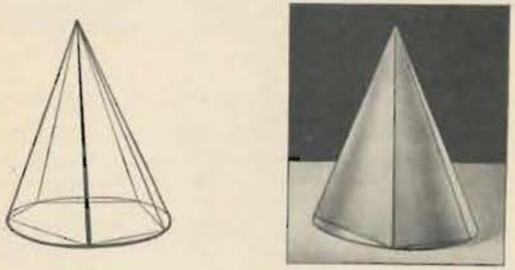


Fig. 221. Cone with Inscribed Pyramid

The base of the inscribed pyramid approaches, both in area and in perimeter, the base of the cone (§ 310); and the altitude of any face of the pyramid approaches an element of the cone, as the pyramid approaches the cone.

If the cone is a right circular cone, the pyramid can be made a regular pyramid; then the slant height of the pyramid approaches the slant height of the cone.

318. Restriction. The word *cone* as used hereafter in this book will be understood to refer to a *circular* cone only. The preceding postulate applies, however, to any kind of cone; and it may be used to obtain results for cones of any form in the manner illustrated below for circular cones only.

We proceed to extend to circular cones certain theorems already proved for pyramids.

319. Theorem XV. *The lateral area* A *of a right circular cone is equal to one half the product of its slant height* l *and the circumference* p *of its base; that is,* $A = l \times p/2$.

Outline of Proof. Inscribe a regular pyramid of n faces in the cone (see Fig. 221); then, by § 298, the lateral area A' of the pyramid is equal to one half the product of its slant height l' and the perimeter p' of its base; that is,

$$A' = \tfrac{1}{2}\, l' \times p';$$

and this formula is correct no matter how large n may be.

By taking n sufficiently large, A' comes to differ by as little as we please from A; while l' and p' come to differ by as little as we please from l and p, respectively. § 317.

Whence, as in § 311,

$$A = \tfrac{1}{2}\, l \times p.$$

320. Corollary 1. *The lateral area of a right circular cone is* $\pi r \cdot l$, *where* r *is the radius of the base and* l *is the slant height.* See §§ 319 and 214.

321. Corollary 2. *The lateral area of a frustum of a right circular cone is equal to one half the product of its slant height and the sum of the circumferences of its bases.*

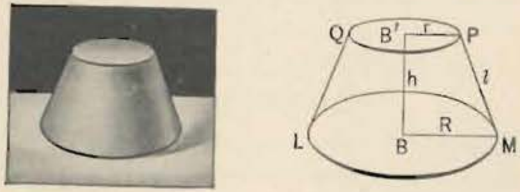


Fig. 222

[The proof is left to the student. Inscribe a frustum of a regular pyramid in the given frustum of a cone, and use § 317 and § 299.]

322. Theorem XVI. *Any section of a circular cone parallel to its base is a circle whose area is to that of the base as the square of its distance from the vertex is to the square of the altitude of the cone.* [HINT. To prove that the section is a circle, pass any two planes through the axis of the cylinder, and show that their intersections with the section are equal. Then inscribe a regular pyramid and proceed as in § 319, using §§ 301 and 317.]

323. Theorem XVII. *The volume V of a cone is equal to one third the product of its base B and its altitude h; that is,* $V = Bh/3$.

[HINT. Use §§ 317, 306, and proceed as in § 319.]

324. Corollary 1. *If from any cone whose base is B and whose altitude is H, a frustum is cut, whose upper base is B' and whose altitude is h, the volume V of the frustum is*

$$V = \frac{BH}{3} - \frac{B'(H-h)}{3}.$$

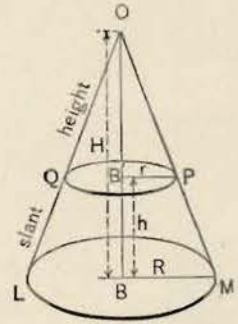


FIG. 223

325. Corollary 2. *The volume of a frustum of any cone is equal to the sum of three cones whose common altitude is the altitude of the frustum and whose bases are the two bases and a mean proportional between them.*

[HINT. Use §§ 322, 324, noting also § 308.]

EXERCISES

1. The altitude of a right circular cone is 12 in. and the radius of the base 9 in. Find the lateral area and the total area of the cone. *Ans.* $424\frac{2}{7}$ sq. in.; $678\frac{6}{7}$ sq. in.

2. How many square yards of canvas are there in a conical tent 12 ft. in diameter and 8 ft. high?

3. The total area of a right circular cone whose altitude is 10 in. is 280 sq. in. Find the total area of the cone cut off by a plane parallel to the base and 6 in. from the vertex.

4. The altitude of a right circular cone is H. How far from the vertex must a plane be passed parallel to the base so that the lateral area of the cone cut off shall be one half that of the original cone? *Ans.* $H/\sqrt{2}$.

[HINT. First prove that the area of the cross section made by the plane will be one half the area of the base. Then apply § 322.]

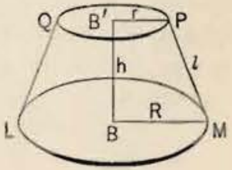
5. The slant height and the diameter of the base of a right circular cone are each equal to l. Find the total area, including the base.

6. The circumference of the base of a circular cone is 11 ft. and its height is 8 ft. What is its volume?

Ans. $\frac{242}{3\pi}$, or $25\frac{2}{3}$ cu. ft.

7. If the height of a circular cone is 10 ft., what must be the radius of its base in order that the volume may be 30 cu. ft.?

8. A frustum of a cone is 1 ft. high and the radii of its bases are respectively 9 ft. and 4 ft. Find its volume.



9. If r and R are the radii of the bases of the frustum of a cone and l is its slant height, find the formula for its volume.

PART IV. GENERAL THEOREMS ON POLYHEDRONS SIMILARITY REGULAR SOLIDS VOLUMES

326. Theorem XVIII. *Two triangular pyramids that have a trihedral angle of the one equal to a trihedral angle of the other are to each other as the products of the edges including the equal trihedral angles.*

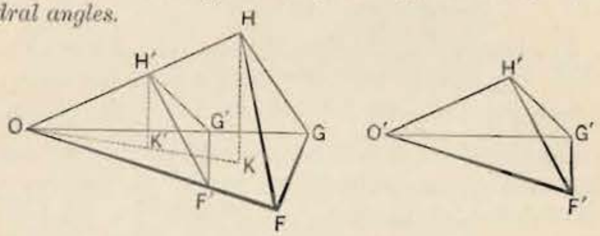


FIG. 224

Given the triangular pyramids $O\text{-}FGH$ and $O'\text{-}F'G'H'$, with the trihedral $\angle O =$ trihedral $\angle O'$, and with volumes denoted by V and V', respectively.

To prove that $V/V' = [OF \cdot OG \cdot OH]/[O'F' \cdot O'G' \cdot O'H']$.

Proof. Place pyramid $O'\text{-}F'G'H'$ so that trihedral $\angle O'$ will coincide with trihedral $\angle O$.

From H and H' draw HK and $H'K'$ perpendicular to the plane OFG.

Then $V/V' = [\triangle OFG \cdot HK]/[\triangle OF'G' \cdot H'K']$

$= [\triangle OFG]/[\triangle OF'G'] \cdot [HK/H'K']$. Why?

But $\dfrac{\triangle OFG}{\triangle OF'G'} = \dfrac{OF \cdot OG}{OF' \cdot OG'}$. § 193

Again, let the plane determined by HK and $H'K'$ intersect plane OFG in line $OK'K$.

Then rt. $\triangle OKH \sim$ rt. $\triangle OK'H'$. Why?

Therefore $HK/H'K' = OH/OH'$. Why?

Therefore $\dfrac{V}{V'} = \dfrac{OF \cdot OG}{OF' \cdot OG'} \cdot \dfrac{OH}{OH'} = \dfrac{OF \cdot OG \cdot OH}{O'F' \cdot O'G' \cdot O'H'}$.

327. Corollary 1. *Two triangular prisms that have a trihedral angle of the one equal to a trihedral angle of the other are to each other as the products of the edges including the trihedral angles.*

[HINT. Break the prism up into triangular pyramids, and use § 326 and Theorem H, § 144.]

328. Corollary 2. *Two parallelepipeds that have a trihedral angle of the one equal to a trihedral angle of the other are to each other as the products of the edges including the trihedral angle.*

329. Similar Tetrahedrons. Two tetrahedrons (that is, triangular pyramids) are said to be **similar** if their faces are similar each to each and similarly placed.

330. Theorem XIX. *The volumes of two similar tetrahedrons are to each other as the cubes of any two corresponding edges.*

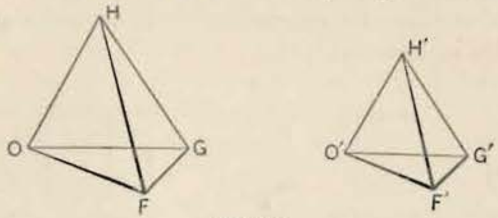


FIG. 225

Given the similar tetrahedrons $O\text{-}FGH$ and $O'\text{-}F'G'H'$ with the volumes denoted by V and V', and with OF and OF' two corresponding edges.

To prove that $V/V' = \overline{OF}^3/\overline{O'F'}^3$. §§ 158, 271

Proof. Trihedral $\angle O =$ trihedral $\angle O'$. § 329

Therefore

$$V/V' = \frac{OF \cdot OG \cdot OH}{O'F' \cdot O'G' \cdot O'H'} = \frac{OF}{O'F'} \cdot \frac{OG}{O'G'} \cdot \frac{OH}{O'H'}. \quad § 326$$

But $OF/O'F' = OG/O'G' = OH/O'H'$. Why?

Therefore

$$V/V' = (OF/O'F')(OF/O'F')(OF/O'F') = \overline{OF}^3/\overline{O'F'}^3.$$

331. Similar Polyhedrons. In general, **similar polyhedrons** are polyhedrons which have the same number of faces similar each to each and similarly placed, and their corresponding polyhedral angles equal.

In the case of similar tetrahedrons, the trihedral angles of the one are necessarily equal to those of the other, if we know only that the faces are similar each to each, since the similarity of the faces makes the three face angles at each vertex equal in the two tetrahedrons, by § 158.

By § 330 and Theorem *H*, § 144, we can show that any two similar polyhedrons are to each other as the cubes of any two corresponding edges.

332. The Regular Solids. A **regular polyhedron** is one whose faces are all congruent regular polygons and whose polyhedral angles are all likewise congruent. Five types of such polyhedrons are represented below.

ICOSAHEDRON DODECAHEDRON OCTAHEDRON CUBE TETRAHEDRON

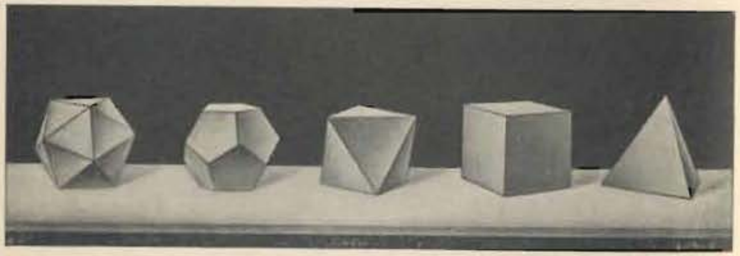


FIG. 226. — THE FIVE REGULAR SOLIDS

We shall now show that the above types are the *only* possible types of regular polyhedrons.

333. Theorem XX. *There exist only five different types of regular polyhedrons.*

Proof. The proof is based upon two facts: (1) that any polyhedral angle has at least three faces and (2) that the sum of the face angles of any convex polyhedral angle must be less than 360°. (See § 273.)

Suppose first that each face is to be a triangle. Then, from the definition of a regular polyhedron, the triangle must be equilateral. Each of its angles will therefore be 60°. Consequently, by statement (2) above, polyhedral angles may be formed by combining three, four, or five such angles, but no more than five can be thus used, since six such angles amount to 360°, while seven or more of them exceed 360°.

Therefore, not more than three regular polyhedrons are possible having triangles as faces. The three that *are* possible are the regular tetrahedron, regular octahedron, and regular icosahedron. (See Fig. 226.)

Suppose secondly that each face is to be a square. Each face angle must then be 90° and, the sum of four such angles being 360°, it follows that but one regular polyhedron is possible having squares as sides. The cube is the one that *is* possible.

Thirdly, suppose that each face is to be a regular pentagon. Since each of the angles of such a figure is 108°, it follows that no more than one regular polyhedron is possible whose faces are pentagons. The one that *is* possible is the dodecahedron.

We can proceed no farther, for the sum of three angles of a regular hexagon is 360°, while the sum of three angles of any regular polygon of more than six sides is greater than 360°.

Hence the theorem is proved.

NOTE. These regular solids occur in nature in the forms of a variety of crystals; but not all crystals are regular solids.

334. Models. Models of the five possible regular polyhedrons can be easily constructed as follows:

Draw diagrams on cardboard as indicated in the figures below. Cut these out and then cut half way through the dotted lines so as to make it easy to fold along these lines.

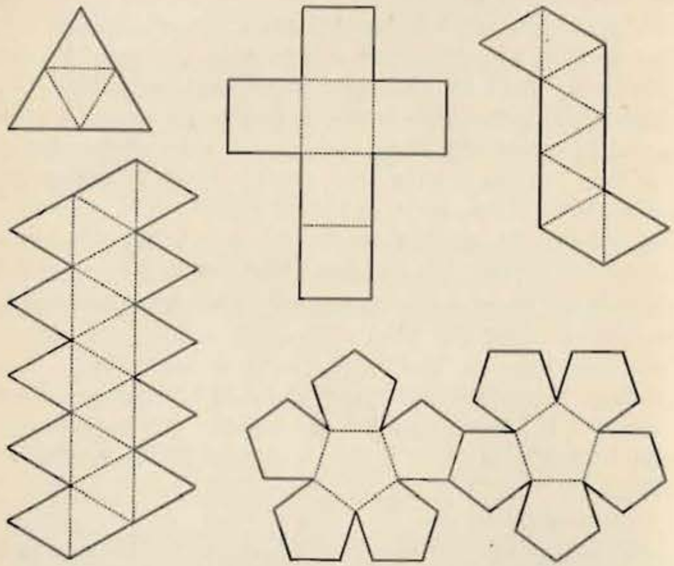


Fig. 227

Fold on the dotted lines so as to bring the edges together, subsequently pasting strips of paper over the edges to hold the solid in position. Models of the tetrahedron, octahedron, and icosahedron may also be made very quickly by hinging together short umbrella wires by means of strong copper wires strung through the end holes, joining together at each corner the proper number of rods. The student may show that each of these models will be quite rigid when completed.

335. Theorem XXI. Cavalieri's Theorem. *If two solids are included between the same pair of parallel planes, and if every section of one of the solids by any plane parallel to one of these parallel planes is equal in area to the section of the other solid by the same plane, the volumes of the two solids are equal.*

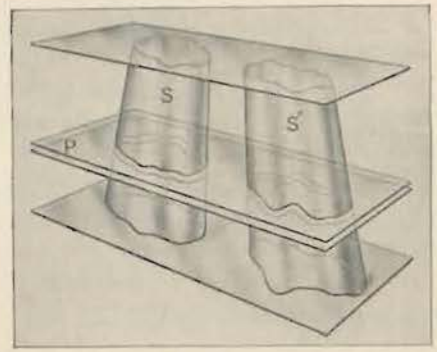


Fig. 228

Outline of Proof. The two solids may be divided into a large number of thin slices by sections parallel to the two including planes. These slices may be thought of as approximately cylindrical, and the sum of all the slices in either case is the volume of the solid.

The bases of two corresponding slices of the two solids between the same two planes are equal in area, by hypothesis. It is therefore apparent that the *volumes* of the two corresponding slices differ as little as we please if their thickness is sufficiently small. It can be shown in a precise manner that the total volume V of one of the solids, which is the sum of all such slices, differs from the total volume V' of the other solid by as little as we please.

Hence, as in §§ 303, 311, it follows that $V = V'$.

Note. Observe also that §§ 293, 303 are essentially special cases of what precedes.

336. Theorem XXII. The Prismoid Formula. *If any solid S of any of the kinds considered in this Chapter is bounded by two parallel plane sections B and T, and if M denotes the area of another section parallel to and midway between B and T, the volume V of S is given by the formula:*

$$V = (B + T + 4\,M) \cdot h/6,$$

where B, T, and M denote the areas of the sections, and h denotes the distance between B and T.

The proof of the preceding formula consists in showing that it reduces in every case to the very formula for volume that has already been proved in the articles above.

Outline of Proof for Prisms and Cylinders. In these cases, all parallel sections are equal (§§ 276, 310). Hence $B = T = M$; and the formula to be proved becomes $V = B \cdot h$, which we have already proved to be correct (§§ 296, 314).

Outline of Proof for Pyramids and Cones. In these cases we know that the area of any section parallel to the base B is proportional to the square of its distance from the vertex (§§ 301, 322). Hence, since M is at a distance $h/2$ from the vertex,

$$\frac{M}{B} = \frac{(h/2)^2}{h^2} = \frac{1}{4}, \text{ or } B = 4\,M.$$

The top section T is zero, since the top bounding plane meets a pyramid or a cone in just one point on the vertex.

Hence the formula to be proved becomes, in this case,

$$V = [B + T + 4\,M] \cdot h/6 = (B + 0 + B) \cdot h/6 = Bh/3,$$

which we know to be correct. §§ 306, 323

Outline of Proof for Frustums. Given a frustum of a pyramid or of a cone, let H be the distance from the vertex O to the larger of the two bounding sections. Let B represent this larger section. Then $H - h$ is the distance from O to the other bounding section T, and $H - h/2$ is the distance from O to the middle section M.

We know that the volume V of the frustum is

$$V = [BH - T(H - h)]/3. \qquad \text{§§ 307, 324}$$

Or, since $\quad T/B = (H - h)^2/H^2, \qquad$ §§ 301, 322

we know that

$$V = B[H - (H - h)^3/H^2]/3 = \frac{B}{H^2}[3\,H^2 - 3\,Hh + h^2]h/3.$$

Since $T/B = (H - h)^2/H^2$, and $M/B = (H - h/2)^2/H^2$, the formula *to be proved* may be written,

$$V = [B + T + 4\,M]h/6 = \left[B + \frac{(H-h)^2}{H^2}B + 4\,\frac{(H - h/2)^2}{H^2}B\right]\frac{h}{6}$$

$$= \frac{B}{H^2}[3\,H^2 - 3\,Hh + h^2] \cdot \frac{h}{3}.$$

This is equivalent to the formula that we know to be correct; hence the theorem is proved.

337. Uses of the Prismoid Formula. The prismoid formula is a convenient means of remembering the volumes of a variety of solids. We shall see in Chapter VIII that it holds for spheres and frustums of spheres as well as for the solids of this chapter.

It also holds for any solid bounded by two parallel planes, made up by joining together pyramids, prisms, etc., bounded by the same two planes; such a solid is called a **prismoid**.

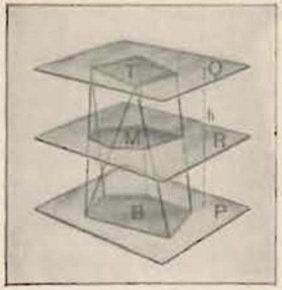


FIG. 229

The formula is used very extensively by engineers to estimate the volumes of various objects, such as the volume of a hill, or the volume of a metal casting. Since the same formula holds for such a large variety of solids, it is reasonably safe to use it *without even stopping to see which of these solids really resembles the object whose volume is desired.*

It is shown in more advanced books that *the same formula holds whenever the area of the section by any plane parallel to the bounding planes is proportional to the square of the distance from some fixed point to that plane.* Many solids not mentioned otherwise in elementary geometry satisfy this requirement.

MISCELLANEOUS EXERCISES. CHAPTER VII

1. Show that every section of a cylinder made by a plane passing through an element is a parallelogram. What is the section when the cylinder is a right cylinder?

2. Show that every section of a cone made by a plane through the vertex is a triangle. What is the section when the cone is a right cone?

3. If a, b, and c are the dimensions of a parallelepiped, show that the length of its diagonal is $\sqrt{a^2+b^2+c^2}$.

4. How long an umbrella will go into a trunk measuring 32 in. by 17 in. by 21 in., inside measure, (a) if the umbrella is laid on the bottom? (b) if it is placed diagonally between opposite corners of the top and bottom?

5. Find the volume of a pyramid whose base is a rhombus 6 in. on a side and whose height is 6 in., if one angle of the rhombus is 60°.

6. The Great Pyramid in Egypt is about 480 ft. high and its base is a square measuring about 764 ft. on a side. Find approximately its volume in cubic yards.

7. Water is poured into a cylindrical reservoir 25 ft. in diameter at the rate of 300 gallons a minute. Find the rate (number of inches per minute) at which the water rises in the reservoir (1 gal. = 231 cu. in.).

8. A copper teapot is $9\frac{7}{8}$ in. in diameter at the bottom, 8 in. at the top, and 11 in. deep. Allowing 42 sq. in. for locks and waste, how much metal is required for its construction, excluding the cover?

9. A conical spire has a slant height of 60 ft. and the perimeter of the base is 50 ft. Find the lateral surface.

10. How many cubic inches of lead are there in a piece of lead pipe 2 yd. long, the outer diameter being 2 in. and the thickness of the lead being $\frac{1}{4}$ of an inch?

11. The chimney of a factory has approximately the shape of a frustum of a regular pyramid. Its height is 75 ft. and its upper and lower bases are squares whose sides are 5 ft. and 8 ft. respectively. The flue is throughout a square whose side is 3 ft. How many cubic feet of material does the chimney contain? Assuming that a brick is 8 in. long, $3\frac{3}{4}$ in. wide, and $2\frac{1}{4}$ in. thick (as is ordinarily the case), estimate the number of bricks in such a chimney.

12. Compare the lateral areas, the total areas, and the volumes of (1) a right circular cylinder and a right circular cone having equal bases and altitudes, (2) a regular pyramid and a regular prism having congruent bases and equal altitudes.

13. A standard rain-gauge is made by inclosing a tube B in the interior of a can $ACDE$ and connecting the mouth of the tube to the mouth of the can by a funnel $FGHI$. The amount of water, measured in inches (depth), that has fallen in the vicinity of the gauge is determined by reading the height of the water in the tube B. Find a formula for the amount of rain that has fallen in terms of the height h of the water in the tube B, the radius r of the tube, and the radius R of the can. *Ans.* hr^2/R^2.

14. If one of the edges of a tetrahedron is 1 in. long, how long will be the corresponding edge of a similar tetrahedron of 8 times the volume? Answer the same question for the case in which the new tetrahedron is to have *half* the volume of the original. *Ans.* 2 in.; $1/\sqrt[3]{2} = .79$ in.

15. It is usual to state the diameter d of a tube in inches, and the area A of its surface in square feet. Show that the formula used by engineers:

$$A = 0.2618\, dl$$

gives very nearly the correct value in square feet, if d is measured in inches, and the length l is measured in feet.

CHAPTER VIII

THE SPHERE

PART I. GENERAL PROPERTIES

338. Spheres. A **sphere** is a portion of space bounded by a surface such that all straight lines to it from a fixed point within are equal.

The fixed point within the sphere is called its **center**; a line segment joining the center to any point on the surface is

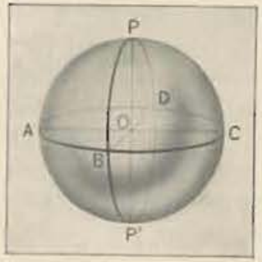


FIG. 230

a **radius**; a line segment drawn through the center and terminated at both ends by the surface is a **diameter**.

It follows from these definitions that:

(*a*) *The radii of a sphere, or equal spheres, are equal.*

(*b*) *The diameters of a sphere, or of equal spheres, are equal.*

(*c*) *Spheres having equal radii, or equal diameters, are equal.*

(*d*) *A sphere may be generated by the revolution of a semicircle about its diameter.*

EXERCISES

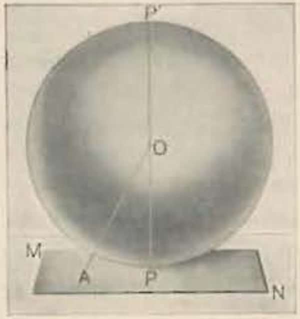
1. What is the locus of the points that are 2 in. from the surface of a sphere whose radius is 4 in.?

2. Show that the distance from the center of a sphere to a point outside the sphere is greater than the radius. (Use Ax. 10.) State the converse. Is it true?

3. If two spheres have the same center, they are called **concentric**. Show that one of two concentric spheres lies wholly within the other.

4. Show that if the center of each one of two given spheres lies on the surface of the other, their radii are equal.

5. Show by § 77, that a plane perpendicular to a diameter of a sphere at its extremity has only one point in common with the sphere.



339. Tangent Planes and Lines. A plane that has only one point in common with a sphere is called a **tangent plane** to the sphere. A line that has only one point in common with a sphere is called a **tangent line** to the sphere. In either case, the single common point is called the **point of tangency**.

340. Theorem I. *A plane perpendicular to a diameter of a sphere at one of its extremities is tangent to the sphere.*

Outline of Proof. Let MN be a plane perpendicular to a diameter PP' at P. Connect any point A of MN to the center of the sphere O. Show, by § 77, that $OA > OP$; whence, by § 338, A cannot be on the sphere, so that P is the *only* point of the plane on the sphere.

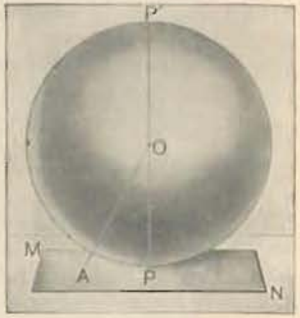


Fig. 231

341. Corollary 1. *(Converse of Theorem I.) If a plane is tangent to a sphere, it is perpendicular to the radius drawn to the point of contact.*

[Hint. Show, by Ax. 10, that the radius is shorter than any other line drawn from the center to the plane. Then use § 77.]

342. Corollary 2. *A straight line perpendicular to a diameter of a sphere at one of its extremities is tangent to the sphere; and conversely.* [Hint. Use §§ 254, 340 for direct, and § 116 for converse.]

343. Corollary 3. *All of the straight lines tangent to a sphere at a given point lie in the plane tangent to the sphere at that point.* [Hint. Use § 254.]

EXERCISES

1. What is the locus of a point in space at a given distance from a given point?

2. Prove that if two lines are tangent to a sphere at the same point, their plane is tangent to the sphere.

[Hint. Connect this with one of the corollaries on this page.]

3. All lines tangent to a sphere from the same point are equal.

[Hint. Connect the center of the sphere with the given point and with two or more points of tangency.]

344. Theorem II. *Every section of a sphere made by a plane is a circle.*

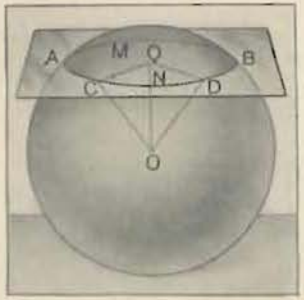


Fig. 232

Given the sphere whose center is O, cut by a plane in the section $AMBN$.

To prove that section $AMBN$ is a circle.

Proof. Draw $OQ \perp$ section $AMBN$; join Q to C and D, any two points in the perimeter of the section; draw OC and OD.

In the rt. $\triangle$ OQC and OQD,

$OQ = OQ$, and $OC = OD$. Why?

Therefore $\triangle OQC \cong \triangle OQD$. Why?

Therefore $QC = QD$. Why?

Since C and D are *any* two points on the perimeter of the section, all points on the perimeter of the section are equally distant from Q. Therefore section $AMBC$ is a circle. § 103

EXERCISES

1. If, in Fig. 232, the radius of the sphere, OC, is 10 in., and the distance OQ from the center O to the plane AB is 6 in., find the radius CQ of the circle.

2. If, in Fig. 232, the distances CQ and OC are given, show how to find the distance OQ.

345. Great and Small Circles. A circle on the sphere whose plane passes through the center is called a **great circle** of the sphere, as CD, Fig. 233.

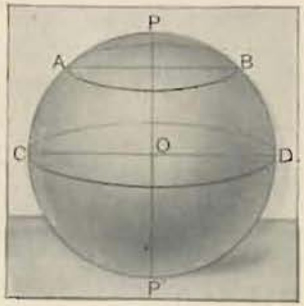


Fig. 233

A circle on the sphere whose plane does not pass through the center is called a **small circle** of the sphere, as AB, Fig. 233.

The **axis** of a circle of a sphere is the diameter of the sphere which is perpendicular to the plane of the circle.

The **poles** of a circle of a sphere are the extremities of the axis of the circle.

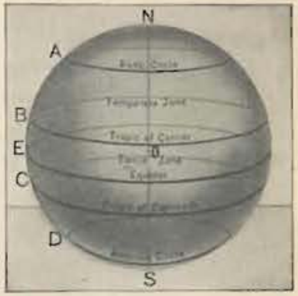
346. Corollary 1. *Through any three points on the surface of a sphere one and only one circle of the sphere may be drawn.*

[Hint. Use 4, § 241.]

347. Corollary 2. *Through any two points on the surface of a sphere a* ***great*** *circle may be drawn.* It follows from 4, § 241, that there is *one and only one* such great circle through the *two* given points, unless they lie at the opposite ends of a diameter.

348. Distance on a Sphere. By the **distance between two points** on the surface of a sphere is meant the length of the shorter arc of the great circle joining them. It can be shown that this is the shortest path on the surface of the sphere between the two points.

EXERCISES



1. If we consider the earth as a sphere, what kind of circles are the parallels of latitude? the equator? the meridians?

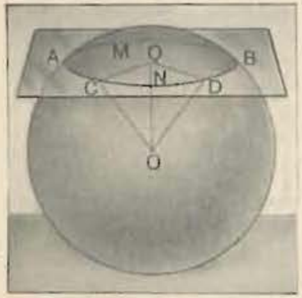
2. Prove that the axis of a small circle of a sphere passes through the center of the circle; and conversely, a diameter of the sphere through the center of a small circle is the axis of that small circle.

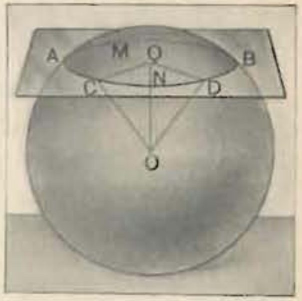
3. Prove that in the same sphere, or in equal spheres, all great circles are equal.

4. The radius of a sphere is 10 in. Find the area of a section made by a plane 5 in. from the center.

5. The area of a section of a sphere 7 in. from the center is $288\,\pi$ sq. in. Find the area of a section 4 in. from the center.

6. Prove that in the same sphere, or in equal spheres, if two sections are equal, they are equally distant from the center, and conversely.





7. Prove that any two great circles of a sphere bisect each other.

349. Theorem III. *All points in the circumference of a circle of a sphere are equally distant from either one of its poles.*

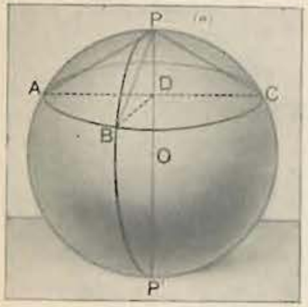


FIG. 234 (*a*)

FIG. 234 (*b*)

Given any two points A and B in the circumference of the circle ABC, and P and P', the poles of ABC.

To prove that $\widehat{PA} = \widehat{PB}$, and $\widehat{P'A} = \widehat{P'B}$.

Proof. Draw the great circles PAP' and PBP'.

Let D be the intersection of the axis PP' with the plane ABC. Draw the straight lines AD, BD, PA and PB.

Now	$PD = PD$, and $DA = DB$,	Why?
and	$\angle PDA = \angle PDB = 90°$.	Why?
Hence	chord PA = chord PB,	Why?
Therefore	$\widehat{PA} = \widehat{PB}$.	Why?

In the same way it may be proved that $\widehat{P'A} = \widehat{P'B}$.

NOTE. The manner in which circles may be drawn on a sphere is illustrated by Fig. 234 (*b*). If one end of a string is held at any point on the sphere, while a pencil attached to the other end is moved around the sphere, keeping the string taut, the end of the pencil describes a circle on the sphere, by Theorem III.

The various figures drawn in this chapter can be reproduced on the surface of an actual sphere, by this method of drawing the circles.

350. Polar Distance. The **polar distance** of a circle of a sphere is the distance on the sphere (§ 348) from its nearest pole to any point of the circumference, as $\widehat{PA}$ or $\widehat{PB}$ in Fig. 234.

A **quadrant** is one fourth part of the circumference of a great circle; *i.e.* an arc of 90° on a great circle.

351. Corollary 1. *The polar distance of a great circle is a quadrant.*

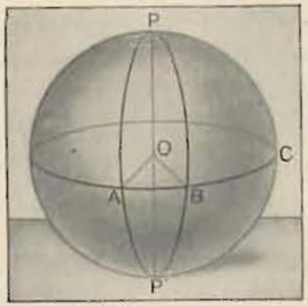


FIG. 235

[HINT. Let ABC be a great circle. Then its center O is also the center of the great circle PBP'. Hence the arc PB measures the right angle POB.]

EXERCISES

1. What is the locus of all the points on the surface of the earth at a quadrant's distance from the north pole? from the south pole? from the equator? See the figure for Ex. 1, p. 289.

2. The distance of the plane of a certain small circle from the center of a sphere is one half the radius of the sphere. If the diameter of the sphere is 12 in., find the polar distance of the small circle in degrees and in inches. *Ans.* 60°; 2π in.

3. Show that a great circle on the earth whose poles lie on the equator passes through the north pole.

352. Problem I. *To determine the radius of a given material sphere.*

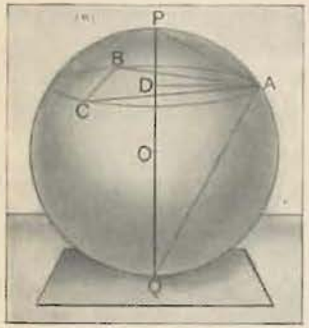


FIG. 236 (a)

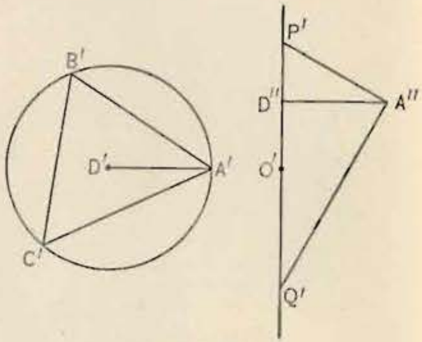


FIG. 236 (b)

Given any material sphere, OPQ.

To find its radius.

Construction. Take any point P on the surface of the sphere as a pole, and describe the circle ABC.

Take any three points on this circle, as A, B, C.

By means of the compasses construct on paper or on the blackboard the triangle $A'B'C'$ congruent to the triangle ABC.

Circumscribe a circle around $\triangle A'B'C'$, and let D' be the center of this circle.

Draw $D''A''$ equal to the radius $D'A'$.

Through D'' draw an indefinite line $P'Q'$ perpendicular to $D''A''$.

From A'' lay off with compasses $A''P'$ equal to line AP.

At A'' erect a perpendicular to $A''P'$ and extend it to meet $P'Q'$ at Q'.

Then $P'Q'$ is the diameter and $P'Q'/2$ is the radius of the given sphere.

[The proof is left to the student.]

353. Problem II. *To construct a sphere through four given points not all in the same plane.*

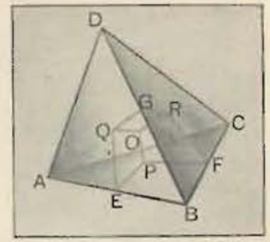


FIG. 237

Given the four points A, B, C, D not all in the same plane.

To construct a sphere that passes through A, B, C, and D.

Construction. At E, the middle point of AB, erect a plane QEP perpendicular to AB. Likewise, let PFR be a plane perpendicular to BC at its middle point F; and let QGR be a plane perpendicular to BD at its middle point G.

Let O be the point common to all three planes QEP, PFR, and QGR.

With O as center, and OA as radius, draw a sphere.

This is the required sphere passing through A, B, C, and D.

Proof. The plane QEP is the locus of all points equidistant from A and B. Why?

Likewise, PFR is the locus of points equidistant from B and C; and QGR is the locus of points equidistant from B and D.

The planes QEP and PFR meet in a line OP. Why?

The line OP meets the plane QGR in a single point O. Why?

Therefore O is equidistant from A, B, C, D. Why?

Moreover, O is the only point equidistant from A, B, C, D. Why?

354. Inscribed and Circumscribed Spheres. A sphere is said to be **circumscribed** about any polyhedron when the vertices of the polyhedron all lie on its surface.

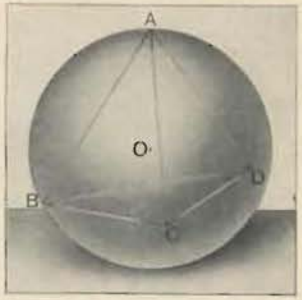


FIG. 238. CIRCUMSCRIBED SPHERE

A sphere is said to be **inscribed** in any polyhedron when it is tangent to each of the faces of the polyhedron.

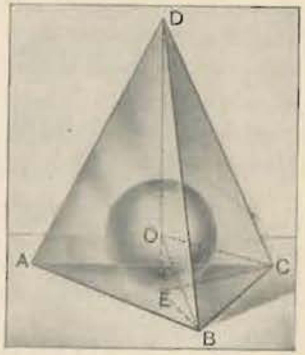


FIG. 239. INSCRIBED SPHERE

355. Corollary 1. *One and only one sphere may be circumscribed about any given tetrahedron (triangular pyramid).*

[HINT. Pass a sphere through the four vertices as in § 353. Notice that the four vertices cannot all lie in one plane.]

356. Problem III. *To inscribe a sphere in a given tetrahedron (triangular pyramid).*

Given the tetrahedron $ABCD$, Fig. 239.

To construct the sphere inscribed in it.

Construction and Proof. Bisect the dihedral angles whose edges are BC, CD, and DB, by the planes BOC, CED, and DEB, respectively.

The plane BOC is the locus of the points equidistant from the faces BCD and BAC; the plane CED is the locus of the points equidistant from the faces BCD and CAD; and the plane DEB is the locus of the points equidistant from the faces BCD and DAB. Why?

The intersection O of these planes is equidistant from the four faces of the tetrahedron. Hence the sphere whose center is O and whose radius is the perpendicular distance OE from O to the face ABC, is tangent to each of the faces; it is therefore inscribed in the tetrahedron.

No other sphere exists that is inscribed in the tetrahedron, for no other point than O is equidistant from the four faces. Why?

EXERCISES

1. By means of an instrument called a spherometer, the distances AD and DP, Fig. 236, can be measured directly. Show, by § 162, how to find the radius from these values.

2. Show that the process of § 352 can be used to find the radius of a sphere, if only a *piece* of the sphere is available, as in the case of a glass lens.

3. Show that four points in space determine a sphere, provided they do not lie in one plane.

4. Show that a sphere is determined if any circle that lies on it and one pole of that circle are given.

5. Show that any two circles of a sphere completely determine the sphere.

PART II. SPHERICAL ANGLES — TRIANGLES — POLYGONS

357. Spherical Angles. The line tangent to a great circle of a sphere at any point is a tangent to the sphere at that point; for it touches the sphere in only one point (§ 339).

The angle formed by the intersection of two great circles is called a **spherical angle**. It is defined to be equal to the angle formed by the tangents to the two great circles, at their point of intersection, as the angle CPD, Fig. 240.

358. Theorem IV. *The angle between two great circles is measured by the arc of a great circle described from its vertex as a pole and included between its sides.*

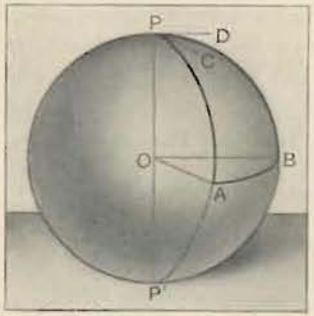


Fig. 240

Given the great circles PAP' and PBP' intersecting at P, and AB the arc of a great circle described with P as a pole.

To prove that $\widehat{AB}$ is the measure of $\angle APB$.

Proof. Draw the radii OA, OB, and the tangents PC, PD.

Then $OA \parallel PC$ and $OB \parallel PD$. Why?

Hence $\angle AOB = \angle CPD$. Why?

But $\angle AOB$ is measured by the arc AB; hence $\angle CPD$ is

measured by the arc AB. It follows that the spherical angle APB, which is equal to $\angle CPD$ by definition (§ 357), is measured by the arc AB.

359. Corollary 1. *The spherical angle between two great circles is equal to the plane angle of the dihedral angle formed by the planes of the two great circles.*

360. Spherical Triangles and Polygons. A **spherical polygon** is a portion of a spherical surface bounded by three or more arcs of **great circles**; as $ABCDE$, Fig. 241.

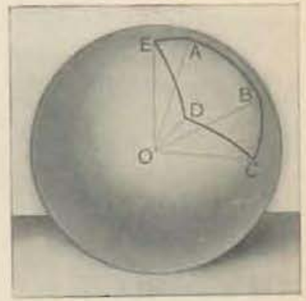


Fig. 241

The bounding arcs of great circles are called the **sides** of the spherical polygon; their intersections, the **vertices**; and the angles formed by the sides at the vertices, the **angles** of the spherical polygon.

A **diagonal** of a spherical polygon is an arc of a great circle joining any two non-adjacent vertices.

A **spherical triangle** is a spherical polygon of three sides, as ABC, Fig. 242.

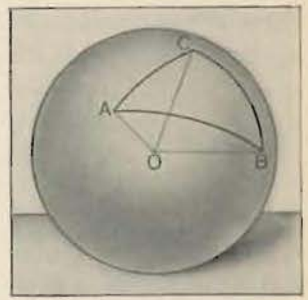


Fig. 242

The words **isosceles, equilateral, acute, right,** and **obtuse** are applied to spherical triangles in precisely the same way as to plane triangles.

Thus, in Fig. 242, the spherical triangle ABC is isosceles if the two sides, as AB and BC, are equal; the triangle is equilateral if $AB = BC = AC$; the triangle is a right triangle if any one angle is a right angle; etc.

361. Relation to Central Polyhedral Angles. The planes of the arcs of the great circles forming the sides of a spherical polygon meet at the center of the sphere and form a polyhedral angle, as O–$ABCD$.

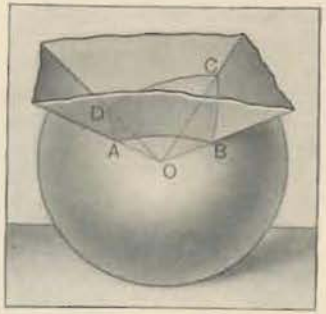


FIG. 243

This polyhedral angle and the spherical polygon are so closely related that the student can easily prove the following statements:

(*a*) *The sides of a spherical polygon have the same measures as the corresponding face angles of the polyhedral angle.*

(*b*) *The angles of the spherical polygon have the same measures as the corresponding dihedral angles of the polyhedral angle.*

Thus, sides AB, BC, etc., of the spherical polygon $ABCD$ have the same measures as face $\angle s$ AOB, BOC, etc., of polyhedral $\angle$ O-$ABCD$; and spherical $\angle s$ ABC, BCD, etc., have the same measures as the dihedral $\angle s$ whose edges are OB, OC, etc.

(*c*) *Any angle of a spherical polygon (or, the corresponding dihedral angle of the polyhedral angle) is measured by the arc of a great circle described with the vertex of the angle as pole and terminated by the sides.* See §§ 358, 359.

In general, *any fact proved for the sides and the angles of a spherical polygon is true also for the corresponding face angles and dihedral angles of the corresponding central polyhedral angle.*

362. Theorem V. *The sum of any two sides of a spherical triangle is greater than the third side.* [Compare § 272.]

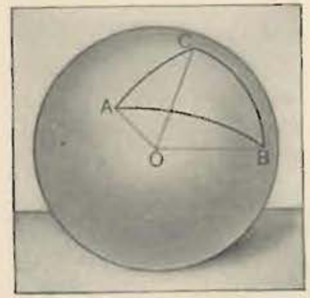


FIG. 244

Given the spherical $\triangle$ ABC.

To prove that $\quad \widehat{AB} + \widehat{BC} > \widehat{CA}$.

Proof. $\quad \angle AOB + \angle BOC > \angle COA$. § 272

$\angle AOB$ is measured by $\widehat{AB}$,

$\angle BOC$ is measured by $\widehat{BC}$,

$\angle COA$ is measured by $\widehat{CA}$. Why?

Therefore $\quad \widehat{AB} + \widehat{BC} > \widehat{CA}$.

363. Theorem VI. *The sum of the sides of any convex spherical polygon is less than 360°.* [Compare § 273.]

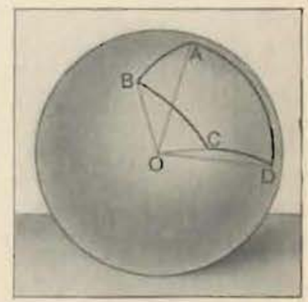


FIG. 245

Given the spherical polygon $ABCD$.

To prove that

$\widehat{AB} + \widehat{BC} + \widehat{CD} + \widehat{DA} < 360°$.

[HINT. Make use of § 273.]

EXERCISES

1. Show that any side of a spherical polygon is less than 180°.

2. In the spherical $\triangle ABC$, $\widehat{AB} = 35°$, and $\widehat{BC} = 75°$. Between what limits must $\widehat{CA}$ lie?

3. Three of the sides of a spherical quadrilateral are respectively 88° 17′, 70° 36′, and 50° 33′. Between what limits must the fourth side lie?

364. Polar Triangles. If from the vertices of a spherical triangle as poles arcs of great circles are drawn, these arcs form a second triangle which is called the **polar triangle** of the first.

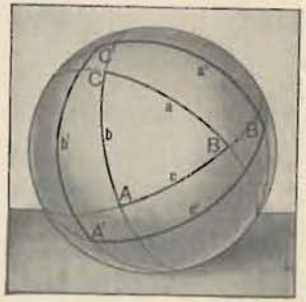


Fig. 246

Thus, if A, B, C, the vertices of the spherical $\triangle ABC$, are the poles of the arcs $B'C'$, $A'C'$, $A'B'$, forming the spherical $\triangle A'B'C'$, then $A'B'C'$ is the polar triangle of ABC.

If the entire circles be drawn, they will intersect so as to form eight spherical triangles, but the polar of the given triangle ABC is that one of the eight triangles whose vertices lie on the same side of the arcs of the given triangle as the corresponding vertices of the given triangle, and no side of which is greater than 180°.

365. Theorem VII. *If one spherical triangle is the polar of another, then the second is the polar of the first.*

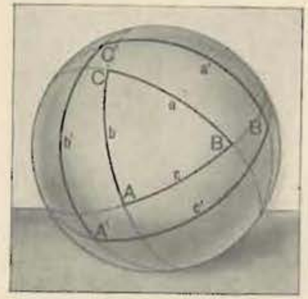


Fig. 247

Given $\triangle A'B'C'$, the polar of $\triangle ABC$.

To prove that $\triangle ABC$ is the polar of $\triangle A'B'C'$.

Proof. A is the pole of $\widehat{B'C'}$, and C is the pole of $\widehat{A'B'}$; Given.

hence B' is at a quadrant's distance from A and C, so that B' is the pole of $\widehat{AC}$. § 249

Similarly, A' is the pole of $\widehat{BC}$, and C' is the pole of $\widehat{AB}$.

Therefore ABC is the polar triangle of $A'B'C'$. § 364.

EXERCISES

1. Show that if one side of a spherical triangle on the earth's surface is on the equator, one vertex of the polar triangle is either at the north pole or at the south pole.

2. Show that if one vertex of a triangle on the earth is at the north pole, one side of the polar triangle is on the equator.

3. Show that if one vertex of a triangle on the earth is at the north pole, and if one side of the triangle is on the equator, the polar triangle also has one vertex at the north pole and one side along the equator.

366. Theorem VIII. *In two polar triangles, each angle of the one is measured by the supplement of the side opposite to it in the other.*

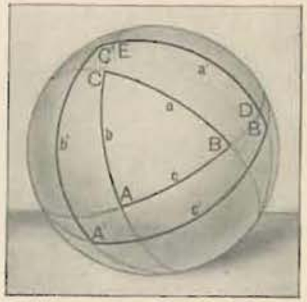


FIG. 248

Given the polar triangles ABC and $A'B'C'$, with the sides denoted by a, b, c, and a', b', c', respectively.

To prove that

(*a*) $\angle A + a' = 180°$, $\angle B + b' = 180°$, $\angle C + c' = 180°$;

(*b*) $\angle A' + a = 180°$, $\angle B' + b = 180°$, $\angle C' + c = 180°$.

Proof. Let $\widehat{AB}$ and $\widehat{AC}$ (prolonged, if necessary) intersect $\widehat{B'C'}$ at D and E, respectively.

Then $\widehat{C'D} = 90°$, and $\widehat{EB'} = 90°$. Why?

Therefore $\widehat{C'D} + \widehat{EB'} = 180°$. Why?

That is $\widehat{C'E} + \widehat{ED} + \widehat{ED} + \widehat{DB'} = 180°$,

or, $\widehat{ED} + a' = 180°$.

But $\widehat{ED}$ is the measure of $\angle A$. (c) § 361

Therefore $\angle A + a' = 180°$.

In a similar manner $\angle B + b' = 180°$, and $\angle C + c' = 180°$.

The proof of (*b*) is left for the student.

EXERCISE

1. If the angles of a spherical triangle are 70°, 90°, and 80°, respectively, find the sides of the polar triangle (in degrees).

367. Theorem IX. *The sum of the angles of a spherical triangle is greater than 180° and less than 540°.*

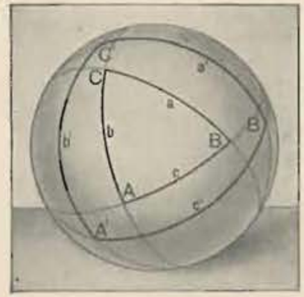


FIG. 249

Given the spherical $\triangle ABC$ with the sides a, b, and c.

To prove that $\angle A + \angle B + \angle C > 180°$ and $< 540°$.

Proof. Let $\triangle A'B'C'$, with its sides denoted by a', b', and c', be the polar of $\triangle ABC$.

Then $\angle A + a' = 180°$, $\angle B + b' = 180°$, $\angle C + c' = 180°$.

Therefore $\angle A + \angle B + \angle C + a' + b' + c' = 540°$. Why?

But $a' + b' + c' < 360°$. § 363

Therefore $\angle A + \angle B + \angle C > 180°$. Why?

Again $a' + b' + c' > 0°$.

Therefore $\angle A + \angle B + \angle C < 540°$. Why?

368. Corollary 1. *In a spherical triangle there can be one, two, or even three right angles; there can be one, two, or three obtuse angles.*

EXERCISES

1. Show that a triangle on the earth's surface whose sides are the equator and two meridians, has two of its angles right angles, and two of its sides quadrants.

2. If, as in Ex. 1, two of the angles of a spherical triangle are right angles, between what limits must the third angle lie?

369. Birectangular and Trirectangular Triangles. A spherical triangle having two right angles is called a **birectangular** spherical triangle.

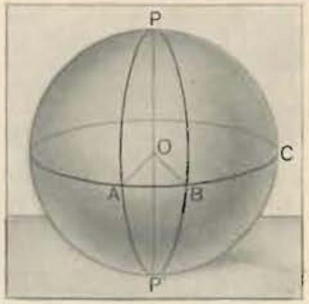


FIG. 250

A spherical triangle having all of its angles right angles is called a **trirectangular** spherical triangle.

If $\angle P$ in the figure is either acute or obtuse, while $\angle s A$ and B are right, $\triangle ABP$ is *birectangular;* if $\angle P$ is also a right angle, $\triangle ABP$ is *trirectangular*, as in Fig. 230, p. 284.

EXERCISES

1. The sides of a spherical triangle are 80°, and 126°, and 175°. How large are the angles of its polar triangle?

2. Show that in a birectangular triangle the sides opposite the right angles are quadrants.

3. Show that three mutually perpendicular planes through the center of a sphere divide its surface into eight congruent trirectangular triangles.

4. Show that the area of a trirectangular triangle on a sphere is one eighth of the area of the sphere.

5. Show that each of the sides of a trirectangular triangle is a quadrant. Hence show that the polar of a trirectangular triangle coincides with it.

370. Symmetric Triangles. Two spherical triangles are **symmetric** when their parts are equal each to each, but are in opposite order. Thus, in the $\triangle s\ ABC$ and $A'B'C'$ (Fig. 251),

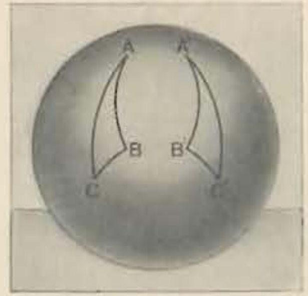


FIG. 251

if angles $A = A'$, $B = B'$, $C = C'$, and sides $AB = A'B'$, $BC = B'C'$, $CA = C'A'$, but the order of arrangement is opposite in the two figures, the triangles are symmetric.

In general, two symmetric triangles cannot be superposed and hence cannot be said to be congruent.

Thus, if $\triangle ABC$ is moved so that side AB coincides with its equal, $A'B'$, in the symmetric $\triangle A'B'C'$, then the vertices C and C' lie on opposite sides of $A'B'$. In plane triangles, $\triangle ABC$ could be revolved about AB till it coincided with $\triangle A'B'C'$; but this is in general impossible with spherical triangles.

371. Corollary. *Two isosceles symmetric spherical triangles are congruent.*

EXERCISES

1. Prove that the base angles of an isosceles spherical triangle are equal.

[HINT. Draw an arc bisecting the vertical angle, thus forming two symmetric triangles.]

2. Show that if two sides of a spherical triangle are quadrants, the triangle is birectangular.

372. Theorem X. *Two triangles on the same sphere, or on equal spheres, are either congruent or symmetric, if two sides and the included angle of the one are equal, respectively, to two sides and the included angle of the other.*

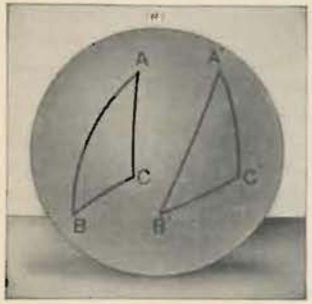


FIG. 252 (*a*)

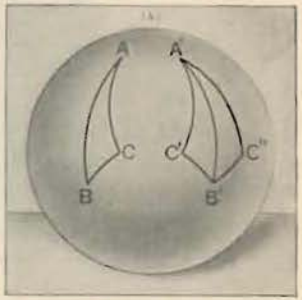


FIG. 252 (*b*)

Given the spherical $\triangle ABC$ and $A'B'C'$ on the same sphere or equal spheres, having $AB = A'B'$, $AC = A'C'$, $\angle A = \angle A'$.

To prove that $\triangle ABC$ and $A'B'C'$ are either congruent or else symmetric.

Proof. If the equal parts of the two triangles are in the same order, $\triangle ABC$ can be placed on $\triangle A'B'C'$ as in the corresponding case of plane triangles. See Fig. 252 (*a*).

If the equal parts of the two triangles are not in the same order, construct $\triangle A'B'C''$ symmetric to $\triangle A'B'C'$. (Fig. 252 (*b*).)

In $\triangle ABC$ and $A'B'C''$, $AC = A'C''$, $AB = A'B'$, and $\angle A = \angle B'A'C''$. Since these parts are arranged in the same order, $\triangle ABC$ and $A'B'C''$ are congruent. Therefore spherical $\triangle ABC$ is symmetric to spherical $\triangle A'B'C'$. Why?

373. Theorem XI. *Two triangles on the same sphere, or on equal spheres, are either congruent or symmetric, if two angles and the included side of the one are equal, respectively, to two angles and the included side of the other.* [Proceed as in § 372.]

374. Theorem XII. *Two triangles on the same sphere, or on equal spheres, are either congruent or symmetric, if the three sides of the one are equal, respectively, to the three sides of the other.*

[The proof is left to the student.]

375. Theorem XIII. *Two triangles on the same sphere, or on equal spheres, are either congruent or symmetric, if the three angles of the one are equal, respectively, to the three angles of the other.*

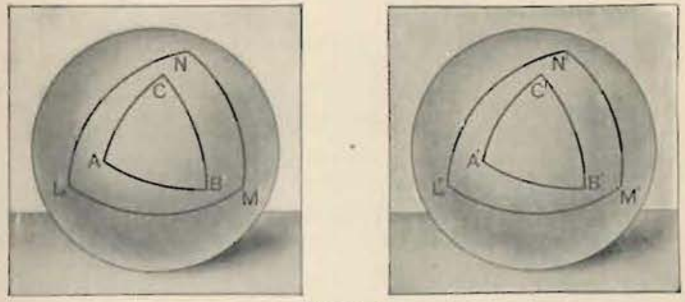


FIG. 253

Outline of Proof. If ABC and $A'B'C'$ are the two given spherical triangles so that $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, their *polar* triangles LMN and $L'M'N'$ have the three *sides* of one equal to the three sides of the other, respectively. § 366

Then, by § 374, $\triangle LMN$ and $L'M'N'$ are either congruent or symmetric. In either case, the three angles of $\triangle LMN$ are equal to the three angles of $\triangle L'M'N'$, respectively; and therefore the three sides of $\triangle ABC$ are equal to the three sides of $\triangle A'B'C'$, respectively. § 366

It follows, by § 374, that $\triangle ABC$ and $A'B'C'$ are either congruent or symmetric.

NOTE. Theorems analogous to those of §§ 41, 43, 44, etc., may be proved in a manner similar to §§ 372–375.

EXERCISES

1. Show that two trihedral angles are congruent if they intercept congruent triangles on the surfaces of two equal spheres whose centers are at their vertices, respectively.

2. If two trihedral angles intercept symmetric spherical triangles on the surface of a sphere whose center is at their vertices, respectively, show that the face angles and the dihedral angles of one trihedral angle are equal to those of the other, but taken in reversed order.

[Such trihedral angles are called **symmetric**.]

3. Prove the following theorem, which states for trihedral angles (§ 361) a theorem analogous to that of § 372:

Two trihedral angles are either congruent or symmetric if two face angles and the included dihedral angle of the one are respectively equal to two face angles and the included dihedral angle of the other.

[HINT: Consider the spherical triangles cut out by the two trihedral angles on the surfaces of two equal spheres whose centers lie at the vertices of the two trihedral angles, and apply § 372.]

4. Prove the following theorem, analogous to § 373:

Two trihedral angles are either congruent or symmetric if two dihedral angles and the included face angle of the one are respectively equal to two dihedral angles and the included face angle of the other.

5. State and prove theorems for trihedral angles similar to Theorems XII–XIII, § 374–375.

6. Prove Theorem XI, § 373, by first considering, as in § 375, the *polars* of the given triangles, and applying § 372.

7. Show that any trirectangular triangle on the earth's surface is congruent to the trirectangular triangle formed by the equator and two meridians whose longitude differs by 90°.

PART III. AREAS AND VOLUMES

376. Theorem XIV. *The area of the surface generated by a straight line revolving about an axis in its plane is equal to the product of the projection of the line on the axis and the length of the circle whose radius is a perpendicular erected at the middle point of the line and terminated by the axis.*

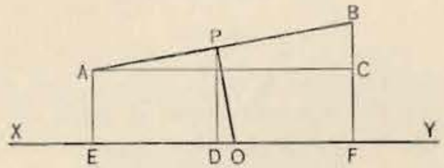


FIG. 254

Given EF, the projection upon XY of AB revolving about XY, and $OP \perp AB$ at its mid-point, and meeting XY at O.

To prove that the area generated by $AB = EF \times 2\,\pi OP$.

Proof. Draw $PD \perp XY$, and $AC \parallel XY$.

Since the surface generated by AB is the lateral surface of the frustum of a cone, the area generated by AB is

$$\frac{AB}{2}\left(2\,\pi AE + 2\,\pi BF\right) = AB \times 2\,\pi \cdot \left(\frac{AE + BF}{2}\right)$$

$$= AB \times 2\,\pi \cdot PD. \qquad \S\ 321$$

Now $\triangle ABC \sim \triangle POD$. § 157

Therefore $AB : OP = AC : PD$. Why?

Then $AB \times PD = AC \times OP = EF \times OP$. Why?

And $AB \times 2\,\pi PD = EF \times 2\,\pi OP$.

That is, the area generated by AB is $EF \times 2\,\pi OP$.

If AB meets XY, the surface generated is a conical surface whose area again $= EF \times 2\,\pi OP$. § 320

If AB is parallel to XY, the surface generated is a cylindrical surface whose area again $= EF \times 2\,\pi OP$. § 313

377. Theorem XV. *The area of the surface of a sphere is equal to the product of its diameter by the circumference of a great circle.*

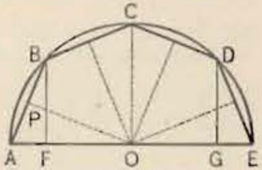


FIG. 255

Given a sphere generated by the revolution of the semicircle $ABCDE$ about the diameter AOE, S being the area of the surface, r being the radius, and d being the diameter.

To prove that $\qquad S = 2\,\pi r d.$

Proof. Inscribe in the semicircle half of a regular polygon $ABCDE$, of any number of sides, and draw BF, CO, DG, perpendicular to AE.

From O draw $OP \perp AB$. Then OP bisects AB, Why?
and is equal to each of the ⊥s drawn from O to the equal chords BC, CD, DE. Why?

Now the area generated by $AB = AF \times 2\,\pi \cdot OP$, § 376
the area generated by $BC = FO \times 2\,\pi \cdot OP$,
the area generated by $CD = OG \times 2\,\pi \cdot OP$,
the area generated by $DE = GE \times 2\,\pi \cdot OP$.

Therefore, if S' denotes the surface generated by the semipolygon,

$$S' = (AF + FO + OG + GE)\,2\,\pi OP = AE \times 2\,\pi OP.$$

Let the number of sides of the semipolygon be now indefinitely increased.

Then OP has for its limit r, the semipolygon for its limit the semicircle, and S' for its limit S. Hence, as in § 303,

$$S = AE \times 2\,\pi r.$$

378. Corollary 1. **The area of the surface of a sphere is equal to $4\,\pi r^2$.**

379. Corollary 2. *The area of the surface of a sphere is equal to the sum of the areas of four great circles.*

For $\qquad S = 2\,r \times 2\,\pi r = 4\,\pi r^2,$ § 378

and πr^2 is the area of a great circle.

380. Corollary 3. *The areas of the surfaces of two spheres are to each other as the squares of their radii; or, as the squares of their diameters.*

381. Zones. A **zone** is a portion of the surface of a sphere bounded by the circumferences of two circles whose planes are parallel.

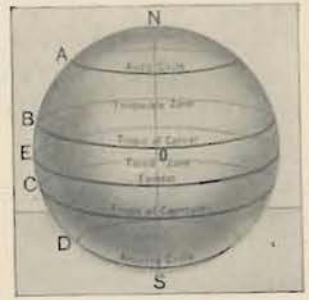


FIG. 256. ZONES ON THE EARTH'S SURFACE

The circumferences forming the boundary of a zone are its **bases**.

If the semicircle NES is revolved about NS as an axis, arc AB will generate a zone, while points A and B will generate the bases of the zone.

The **altitude** of a zone is the perpendicular distance between the planes of the bases.

382. Corollary 4. *The area of a zone of a sphere is equal to the product of the altitude h of the zone and the circumference of a great circle; or $2\,\pi r h$, where r is the radius of the sphere.*

383. Lunes. A **lune** is a portion of a spherical surface bounded by two semicircumferences of great circles; as

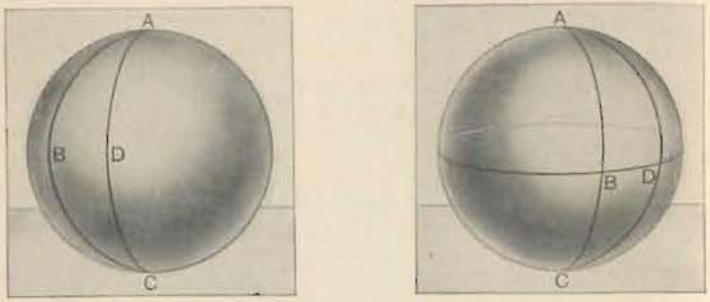


FIG. 257

$ABCDA$ (Fig. 257). The **angle of a lune** is the angle formed by its bounding arcs. Thus BAD is the angle of the lune $ABCDA$.

384. Theorem XVI. *The area of a lune is to the area of the surface of the sphere as the angle of the lune is to four right angles.*

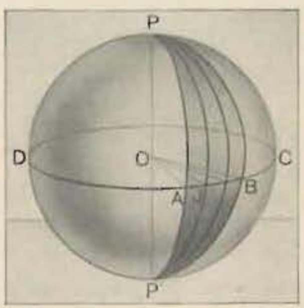


FIG. 258

Given the lune $PAP'B$, let L denote the area of the lune, S the area of the surface of the sphere, and a the angle of the lune.

To prove that $L/S = \angle a/4 \text{ rt. } \angle s$.

Proof. With P as a pole describe the great circle $ABCD$.

Then the arc AB measures $\angle a$ of the lune. Why?

Therefore arc AB/circle $ABCD = \angle a/4 \text{ rt. } \angle s$.

If AB and $ABCD$ are commensurable, let their common measure be contained m times in AB and n times in $ABCD$.

Then arc AB/circle $ABCD = m/n$.

Therefore $a/4 \text{ rt. } \angle s = m/n$. § 358

Pass arcs of great circles through each point of division of $ABCD$ and the poles P and P'.

These arcs will divide the entire surface into n equal lunes, of which $PAP'B$ will contain m.

Therefore $L/S = m/n$,

or, $L/S = a/4 \text{ rt. } \angle s$.

If AB and $ABCD$ are incommensurable, the theorem can be proved as in § 130. The details are left to the student.

385. Corollary 1. *The area of a lune whose angle is* $1°$ *is* $4\,\pi r^2/360 = \pi r^2/90$.

386. Corollary 2. *The area of a lune whose angle is* $k°$ *is* $4\,\pi r^2 k/360 = \pi r^2 k/90$.

EXERCISES

1. If the surface of a sphere is 10 sq. ft., what is the area of a lune whose angle is 40°? What is the radius of the sphere? *Ans.* $1\frac{1}{9}$ sq. ft.; 0.89^{+} ft.

2. Show that two lunes on the same sphere or equal spheres have the same ratio as their angles.

3. What is the angle of a lune which has the same area as a trirectangular triangle?

4. Show that the area of a lune is one ninetieth of the area of a great circle multiplied by the number of degrees in the angle of the lune.

387. Theorem XVII. *Two symmetric triangles are equal in area.*

Given the two symmetric spherical triangles ABC and $A'B'C'$.

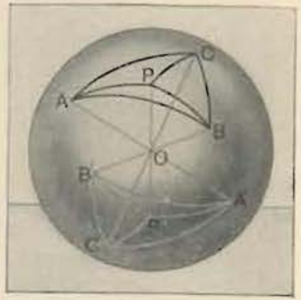


FIG. 259

To prove that

$$\triangle ABC = \triangle A'B'C'.$$

Proof. Let P be the pole of the small circle passing through the points A, B, C, and draw the great circle arcs PA, PB, and PC.

Then $PA = PB = PC$. Why?

Now place the two triangles diametrically opposite to each other and draw the diameter POP'. Also draw the great circle arcs $P'A'$, $P'B'$, and $P'C'$. Then the triangles PBC and $P'B'C'$ are symmetrical and isosceles and therefore congruent. § 371.

Similarly $\triangle PCA \cong \triangle P'C'A'$,

and $\triangle PAB \cong \triangle P'A'B'$.

That is, the three parts of ABC are respectively congruent to the three parts of ABC.

Therefore $\triangle ABC = \triangle A'B'C'$.

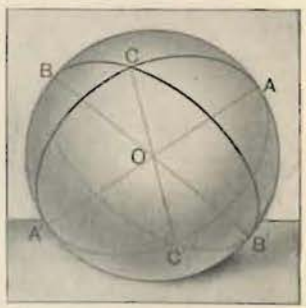


FIG. 260

388. Corollary 1. *If two semicircumferences of great circles BCB' and ACA' intersect on the surface of a hemisphere, the sum of the areas of the two opposite spherical triangles ACB and $A'CB'$ is equal to the area of a lune whose angle is equal to ACB.*

[HINT. Show that the triangle ABC is symmetric to the triangle $A'B'C'$. Hence show that $\triangle ACB + \triangle A'CB' =$ lune $A'CB'C'$.]

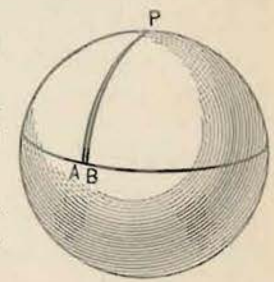
389. Spherical Degree. The area of a lune whose angle is 1° is $4\pi r^2/360$, or $\pi r^2/90$ (§ 385). Half this area, that is, $4\pi r^2/720$ or $\pi r^2/180$, is often *taken as a unit of area on the sphere*, and it is called a **spherical degree**.

390. Measure of Solid Angles. A trihedral angle whose vertex is at the center of a sphere cuts out a spherical triangle on the surface of the sphere. The area of the spherical triangle, in spherical degrees, is called the **measure** of the trihedral angle.

Likewise any polyhedral angle is measured by the area, in spherical degrees, that it cuts out upon the surface of a sphere whose center is at its vertex.

EXERCISES

1. Show that the area of a lune whose angle is 1° is 2 spherical degrees.



2. Show that the area of the entire sphere is 720 spherical degrees.

3. Show that the area of a birectangular triangle whose third angle is 1° is 1 spherical degree.

4. Show that the area of a trirectangular triangle is 90 spherical degrees, or one eighth of the entire surface.

391. Spherical Excess. The excess of the sum of the angles of a spherical triangle over 180° is called the **spherical excess** of the triangle.

If, for example, the angles of a spherical triangle are 80°, 100°, and 125°, the spherical excess of the triangle is 125°.

Likewise, the spherical excess of any spherical polygon is the excess of the sum of its angles above the sum of the angles of a plane polygon of the same number of sides.

392. Theorem XVIII. *The area of a spherical triangle is equal to the area of a lune whose angle is half the spherical excess of the triangle.*

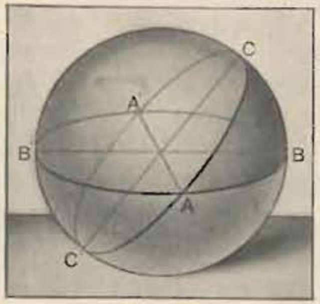


FIG. 261

Given the spherical $\triangle ABC$.

To prove that $\triangle ABC$ is equal to a lune whose angle is

$$\tfrac{1}{2}(\angle A + \angle B + \angle C - 180°).$$

Proof. Complete the great circles by producing the sides of the $\triangle ABC$, as in Fig. 261.

Since $\triangle s\ AB'C'$ and $A'BC$ are symmetric, they are equal in area. § 387

Therefore lune $ABA'C = \triangle ABC + \triangle AB'C'$. § 388

But, denoting the area of the whole sphere by S,

$$\triangle CB'A + \triangle AC'B + \triangle ABC + \triangle AB'C' = \tfrac{1}{2}S. \quad \text{Why?}$$

Therefore

$$(\text{lune } BCB'A - \triangle ABC) + (\text{lune } CAC'B - \triangle ABC)$$
$$+ \text{lune } ABA'C = \tfrac{1}{2}S. \quad \text{Why?}$$

Therefore, transposing, we obtain

$$2\triangle ABC = \text{lune } ABA'C + \text{lune } BCB'A + \text{lune } CAC'B - \tfrac{1}{2}S.$$

But $\frac{1}{2}S$ is the area of a lune whose angle is 180°.

Therefore $\triangle ABC$ is equal to a lune whose angle is

$$\tfrac{1}{2}(\angle A + \angle B + \angle C - 180°). \quad § 384$$

393. Corollary 1. *The area of a spherical triangle, measured in spherical degrees, is numerically equal to its spherical excess.*

NOTE. This result enables us to compute the area of any spherical triangle in ordinary units of area, when we know its angles and the radius of the sphere. Thus, if r denotes the radius of the sphere, E the spherical excess, and A the required area, we have, by § 385

$$A = E \times \frac{\pi r^2}{180} = \frac{\pi r^2 E}{180}.$$

394. Corollary 2. *The area of a trirectangular triangle is 90 spherical degrees.*

395. Corollary 3. *The area of any spherical triangle is to the area of the entire sphere as its spherical excess is to 720°.*

396. Solid Angles. In general, if any closed polygon or curve is drawn on the surface of a sphere, the figure formed by all radii of the sphere that join the center to the points of this figure on the spherical surface is called a **solid angle**. The area on the surface of the sphere cut out by such a solid angle, in spherical degrees, is the **measure** of the solid angle.

EXERCISES

1. What is the measure of a hemisphere in spherical degrees?

2. The radius of a sphere is 2 ft. Find the area of a triangle on its surface whose angles are 75°, 35°, 105°, respectively. Solve first by § 392; then by § 394. *Ans.* $7\pi/9$ sq. ft.

3. The radius of the earth is approximately 4000 miles. Find the entire area. Show that the area in square miles of one spherical degree is approximately 278,000 square miles.

4. Find how large a triangle on the earth's surface would have the total sum of its three angles equal to 181°.

5. Show that a region containing about 270,000 sq. mi. on the earth contains no triangle whose spherical excess is 1°.

6. What is the area of the state in which you live? What is its measure in spherical degrees?

397. Theorem XIX. *The volume V of a sphere is equal to the product of its surface by one third of its radius; or,* $V = 4\pi r^3/3$.

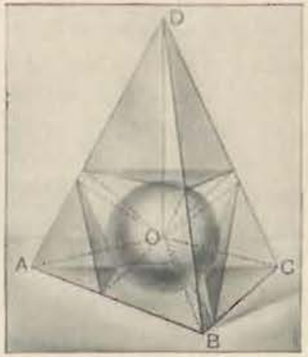


FIG. 262

Given a sphere whose center is O; let S denote its surface, r its radius, and V its volume.

To prove that $V = S \times r/3 = 4\pi r^3/3$

Proof. Circumscribe about the sphere any polyhedron as D-ABC, and denote its surface by S' and its volume by V'.

Form pyramids, as O-ABC, etc., having the faces of the polyhedron as bases and the center of the sphere as a common vertex.

These pyramids will have a common altitude equal to r, and the volume of each pyramid is equal to its *base* $\times r/3$. Why?

Therefore $V' = S' \times r/3$. Why?

If the number of pyramids is indefinitely increased by passing planes tangent to the sphere at points where the edges of the pyramids cut the surface of the sphere, as in Fig. 262, the difference between S and S' becomes as small as we please; the difference between V and V' becomes as small as we please.

But however great the number of pyramids,

$$V' = S' \times r/3.$$

Therefore, as in § 303, $V = S \times r/3$.

Since $S = 4\pi r^2$,

it follows that $V = 4\pi r^2 \times r/3 = 4\pi r^3/3$.

398. Corollary 1. *The volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

399. Corollary 2. *The volume of the pyramidal piece cut out of a sphere by any polyhedral angle whose vertex is at the center is equal to one third the area of the spherical polygon cut out of the surface times the radius.*

400. Corollary 3. *The prismoid formula* (§ 336) *holds for a sphere.*

[HINT. Two parallel planes that include the entire sphere are tangent planes at the ends of a diameter; these cut the sphere in only one point each. A plane parallel to these two and halfway between them cuts the sphere in a great circle. Hence, in the notation of § 336, $B = 0$, $T = 0$, $M = 4\pi r^2$, $h = 2r$; hence the prismoid formula would give

$$V = h\left[\frac{B + T + 4M}{6}\right] = 2r\left[\frac{0 + 0 + 4\pi r^2}{6}\right] = \frac{4\pi r^3}{3},$$

which, by § 397, is correct.]

[NOTE. As a matter of fact, the prismoid formula holds for the portion of a sphere intercepted between any two parallel planes.]

EXERCISES

1. Assuming that the earth is a sphere whose radius is 4000 mi., find its volume.

2. Show that a cube circumscribed about a sphere has a volume $8r^3$. Hence show that the sphere occupies a little more than half this volume

3. Find the volume of the material in a hollow sphere, if the radius of the outer surface is 6 in. and that of the inner surface is 5 in.

4. Show that the volume of a hollow sphere whose outer and inner radii are R and r, respectively, is $4\pi(R^3 - r^3)/3$.

5. Find the volume of the material in a hollow sphere whose outer radius is 10 in., if the material is $\frac{1}{2}$ in. thick.

6. Show that the volume of a sphere in terms of its diameter, d, is $\pi d^3/6$.

7. If the radius of one sphere is twice that of another, how do their volumes compare?

8. If the volume of one sphere is twice that of another, how do their radii compare?

9. Find approximately the radius of a sphere whose volume is 100 cu. in.

10. How many shot $\frac{1}{10}$ in. in diameter can be made from 10 cu. in. of lead?

11. If oranges 3 in. in diameter sell for 30 cents per dozen, and those 4 in. in diameter sell for 50 cents per dozen, which are the cheaper by volume?

12. If the skins are of equal thickness, which of the oranges of Ex. 11 has the greater percentage of skin to the cubic inch of volume?

13. Assuming that raindrops are practically spherical, if the diameter of one drop is half that of another, how do their volumes compare? their areas?

14. Which of the two drops of Ex. 13 has the greater ratio of area to volume? How much greater? Which will fall the more rapidly through the air?

[Hint. The greater the ratio of area to volume, for the same material, the slower the body will fall through the air.]

15. Explain, by the principle of Ex. 14, why very small dust particles remain floating in the air for a long time.

16. The same amount of material, in the form of a cube, is melted and cast into a sphere. Is the surface area less in the form of the cube or in that of the sphere?

[Hint. Assume the cube to be 1 unit on each edge; find the radius of the resulting sphere.]

17. If a surveyor wishes to be certain that the sum of the angles in any triangle in a region on the earth's surface shall be equal to 180° to within one minute, how large may the region be? *Ans.* About 4600 sq. mi.

18. Demonstrate the existence of spherical triangles with three obtuse angles from the existence of triangles whose sides are very short.

TABLES

TABLE I

Ratios of the Sides of Right Triangles
and
Chords and Arcs of a Unit Circle

TABLE II

Squares and Square Roots of Numbers
Cubes and Cube Roots of Numbers

TABLE III

Values of Important Numbers
including
Units of Measurement

TABLE I

RATIOS OF THE SIDES OF RIGHT TRIANGLES

AND

LENGTHS OF CHORDS AND ARCS OF A UNIT CIRCLE

EXPLANATION OF TABLE I

1. Ratios of the Sides of Right Triangles. If an angle given in the ***Angle Column*** is one acute angle of a ***right triangle:***

The Sine Column gives the ratio of the side opposite the angle to the hypotenuse;

The Tangent Column gives the ratio of the side opposite the angle to the side adjacent to the angle.

To find the ***Cosine*** of any angle, take the ***sine of the complement*** of that angle.

2. Chords and Arcs of a Unit Circle. If an angle given in the ***Angle Column*** is an angle at the center of a ***circle of unit radius:***

The Chord Column gives the length of the chord that subtends that angle;

The Arc Column gives the length of the arc that subtends that angle.

To find the ***lengths of chords or arcs of any circle of radius*** r, multiply the values given in the table by that radius.

The table is limited to angles less than 90°; but to find ***the chord that subtends an obtuse angle,*** first take half the angle, find the sine of this half angle, and multiply by 2. This follows from the fact that the chord of any angle is *twice the sine of half that angle.*

Angle	Sine	Tangent	Chord	Arc	Angle	Sine	Tangent	Chord	Arc
0° 00′	.0000	.0000	.0000	.0000	**9° 00′**	.1564	.1584	.1569	.1571
10	.0029	.0029	.0029	.0029	10	.1593	.1614	.1598	.1600
20	.0058	.0058	.0058	.0058	20	.1622	.1644	.1627	.1629
30	.0087	.0087	.0087	.0087	30	.1650	.1673	.1656	.1658
40	.0116	.0116	.0116	.0116	40	.1679	.1703	.1685	.1687
50	.0145	.0145	.0145	.0145	50	.1708	.1733	.1714	.1716
1° 00′	.0175	.0175	.0175	.0175	**10° 00′**	.1736	.1763	.1743	.1745
10	.0204	.0204	.0204	.0204	10	.1765	.1793	.1772	.1774
20	.0233	.0233	.0233	.0233	20	.1794	.1823	.1801	.1804
30	.0262	.0262	.0262	.0262	30	.1822	.1853	.1830	.1833
40	.0291	.0291	.0291	.0291	40	.1851	.1883	.1859	.1862
50	.0320	.0320	.0320	.0320	50	.1880	.1914	.1888	.1891
2° 00′	.0349	.0349	.0349	.0349	**11° 00′**	.1908	.1944	.1917	.1920
10	.0378	.0378	.0378	.0378	10	.1937	.1974	.1946	.1949
20	.0407	.0407	.0407	.0407	20	.1965	.2004	.1975	.1978
30	.0436	.0437	.0436	.0436	30	.1994	.2035	.2004	.2007
40	.0465	.0466	.0465	.0465	40	.2022	.2065	.2033	.2036
50	.0494	.0495	.0494	.0495	50	.2051	.2095	.2062	.2065
3° 00′	.0523	.0524	.0524	.0524	**12° 00′**	.2079	.2126	.2091	.2094
10	.0552	.0553	.0553	.0553	10	.2108	.2156	.2119	.2123
20	.0581	.0582	.0582	.0582	20	.2136	.2186	.2148	.2153
30	.0610	.0612	.0611	.0611	30	.2164	.2217	.2177	.2182
40	.0640	.0641	.0640	.0640	40	.2193	.2247	.2206	.2211
50	.0669	.0670	.0669	.0669	50	.2221	.2278	.2235	.2240
4° 00′	.0698	.0699	.0698	.0698	**13° 00′**	.2250	.2309	.2264	.2269
10	.0727	.0729	.0727	.0727	10	.2278	.2339	.2293	.2298
20	.0756	.0758	.0756	.0756	20	.2306	.2370	.2322	.2327
30	.0785	.0787	.0785	.0785	30	.2334	.2401	.2351	.2356
40	.0814	.0816	.0814	.0814	40	.2363	.2432	.2380	.2385
50	.0843	.0846	.0843	.0844	50	.2391	.2462	.2409	.2414
5° 00′	.0872	.0875	.0872	.0873	**14° 00′**	.2419	.2493	.2437	.2443
10	.0901	.0904	.0901	.0902	10	.2447	.2524	.2466	.2473
20	.0929	.0934	.0931	.0931	20	.2476	.2555	.2495	.2502
30	.0958	.0963	.0960	.0960	30	.2504	.2586	.2524	.2531
40	.0987	.0992	.0989	.0989	40	.2532	.2617	.2553	.2560
50	.1016	.1022	.1018	.1018	50	.2560	.2648	.2582	.2589
6° 00′	.1045	.1051	.1047	.1047	**15° 00′**	.2588	.2679	.2611	.2618
10	.1074	.1080	.1076	.1076	10	.2616	.2711	.2639	.2647
20	.1103	.1110	.1105	.1105	20	.2644	.2742	.2668	.2676
30	.1132	.1139	.1134	.1134	30	.2672	.2773	.2697	.2705
40	.1161	.1169	.1163	.1164	40	.2700	.2805	.2726	.2734
50	.1190	.1198	.1192	.1193	50	.2728	.2836	.2755	.2763
7° 00′	.1219	.1228	.1221	.1222	**16° 00′**	.2756	.2867	.2783	.2793
10	.1248	.1257	.1250	.1251	10	.2784	.2899	.2812	.2822
20	.1276	.1287	.1279	.1280	20	.2812	.2931	.2841	.2851
30	.1305	.1317	.1308	.1309	30	.2840	.2962	.2870	.2880
40	.1334	.1346	.1337	.1338	40	.2868	.2994	.2899	.2909
50	.1363	.1376	.1366	.1367	50	.2896	.3026	.2927	.2938
8° 00′	.1392	.1405	.1395	.1396	**17° 00′**	.2924	.3057	.2956	.2967
10	.1421	.1435	.1424	.1425	10	.2952	.3089	.2985	.2996
20	.1449	.1465	.1453	.1454	20	.2979	.3121	.3014	.3025
30	.1478	.1495	.1482	.1484	30	.3007	.3153	.3042	.3054
40	.1507	.1524	.1511	.1513	40	.3035	.3185	.3071	.3083
50	.1536	.1554	.1540	.1542	50	.3062	.3217	.3100	.3113
9° 00′	.1564	.1584	.1569	.1571	**18° 00′**	.3090	.3249	.3129	.3142

Angle	Sine	Tangent	Chord	Arc	Angle	Sine	Tangent	Chord	Arc
18°00′	.3090	.3249	.3129	.3142	**27°00′**	.4540	.5095	.4669	.4712
10	.3118	.3281	.3157	.3171	10	.4566	.5132	.4697	.4741
20	.3145	.3314	.3186	.3200	20	.4592	.5169	.4725	.4771
30	.3173	.3346	.3215	.3229	30	.4617	.5206	.4754	.4800
40	.3201	.3378	.3244	.3258	40	.4643	.5243	.4782	.4829
50	.3228	.3411	.3272	.3287	50	.4669	.5280	.4810	.4858
19°00′	.3256	.3443	.3301	.3316	**28°00′**	.4695	.5317	.4838	.4887
10	.3283	.3476	.3330	.3345	10	.4720	.5354	.4867	.4916
20	.3311	.3508	.3358	.3374	20	.4746	.5392	.4895	.4945
30	.3338	.3541	.3387	.3403	30	.4772	.5430	.4923	.4974
40	.3365	.3574	.3416	.3432	40	.4797	.5467	.4951	.5003
50	.3393	.3607	.3444	.3462	50	.4823	.5505	.4979	.5032
20°00′	.3420	.3640	.3473	.3491	**29°00′**	.4848	.5543	.5008	.5061
10	.3448	.3673	.3502	.3520	10	.4874	.5581	.5036	.5091
20	.3475	.3706	.3530	.3549	20	.4899	.5619	.5064	.5120
30	.3502	.3739	.3559	.3578	30	.4924	.5658	.5092	.5149
40	.3529	.3772	.3587	.3607	40	.4950	.5696	.5120	.5178
50	.3557	.3805	.3616	.3636	50	.4975	.5735	.5148	.5207
21°00′	.3584	.3839	.3645	.3665	**30°00′**	.5000	.5774	.5176	.5236
10	.3611	.3872	.3673	.3694	10	.5025	.5812	.5204	.5265
20	.3638	.3906	.3702	.3723	20	.5050	.5851	.5233	.5294
30	.3665	.3939	.3730	.3752	30	.5075	.5890	.5261	.5323
40	.3692	.3973	.3759	.3782	40	.5100	.5930	.5289	.5352
50	.3719	.4006	.3788	.3811	50	.5125	.5969	.5317	.5381
22°00′	.3746	.4040	.3816	.3840	**31°00′**	.5150	.6009	.5345	.5411
10	.3773	.4074	.3845	.3869	10	.5175	.6048	.5373	.5440
20	.3800	.4108	.3873	.3898	20	.5200	.6088	.5401	.5469
30	.3827	.4142	.3902	.3927	30	.5225	.6128	.5429	.5498
40	.3854	.4176	.3930	.3956	40	.5250	.6168	.5457	.5527
50	.3881	.4210	.3959	.3985	50	.5275	.6208	.5485	.5556
23°00′	.3907	.4245	.3987	.4014	**32°00′**	.5299	.6249	.5513	.5585
10	.3934	.4279	.4016	.4043	10	.5324	.6289	.5541	.5614
20	.3961	.4314	.4044	.4072	20	.5348	.6330	.5569	.5643
30	.3987	.4348	.4073	.4102	30	.5373	.6371	.5597	.5672
40	.4014	.4383	.4101	.4131	40	.5398	.6412	.5625	.5701
50	.4041	.4417	.4130	.4160	50	.5422	.6453	.5652	.5730
24°00′	.4067	.4452	.4158	.4189	**33°00′**	.5446	.6494	.5680	.5760
10	.4094	.4487	.4187	.4218	10	.5471	.6536	.5708	.5789
20	.4120	.4522	.4215	.4247	20	.5495	.6577	.5736	.5818
30	.4147	.4557	.4244	.4276	30	.5519	.6619	.5764	.5847
40	.4173	.4592	.4272	.4305	40	.5544	.6661	.5792	.5876
50	.4200	.4628	.4300	.4334	50	.5568	.6703	.5820	.5905
25°00′	.4226	.4663	.4329	.4363	**34°00′**	.5592	.6745	.5847	.5934
10	.4253	.4699	.4357	.4392	10	.5616	.6787	.5875	.5963
20	.4279	.4734	.4386	.4422	20	.5640	.6830	.5903	.5992
30	.4305	.4770	.4414	.4451	30	.5664	.6873	.5931	.6021
40	.4331	.4806	.4442	.4480	40	.5688	.6916	.5959	.6050
50	.4358	.4841	.4471	.4509	50	.5712	.6959	.5986	.6080
26°00′	.4384	.4877	.4499	.4538	**35°00′**	.5736	.7002	.6014	.6109
10	.4410	.4913	.4527	.4567	10	.5760	.7046	.6042	.6138
20	.4436	.4950	.4556	.4596	20	.5783	.7089	.6070	.6167
30	.4462	.4986	.4584	.4625	30	.5807	.7133	.6097	.6196
40	.4488	.5022	.4612	.4654	40	.5831	.7177	.6125	.6225
50	.4514	.5059	.4641	.4683	50	.5854	.7221	.6153	.6254
27°00′	.4540	.5095	.4669	.4712	**36°00′**	.5878	.7265	.6180	.6283

Angle	Sine	Tangent	Chord	Arc	Angle	Sine	Tangent	Chord	Arc
36°00′	.5878	.7265	.6180	.6283	**45°00′**	.7071	1.0000	.7654	.7854
10	.5901	.7310	.6208	.6312	10	.7092	1.0058	.7681	.7883
20	.5925	.7355	.6236	.6341	20	.7112	1.0117	.7707	.7912
30	.5948	.7400	.6263	.6370	30	.7133	1.0176	.7734	.7941
40	.5972	.7445	.6291	.6400	40	.7153	1.0235	.7761	.7970
50	.5995	.7490	.6318	.6429	50	.7173	1.0295	.7788	.7999
37°00′	.6018	.7536	.6346	.6458	**46°00′**	.7193	1.0355	.7815	.8029
10	.6041	.7581	.6374	.6487	10	.7214	1.0416	.7841	.8058
20	.6065	.7627	.6401	.6516	20	.7234	1.0477	.7868	.8087
30	.6088	.7673	.6429	.6545	30	.7254	1.0538	.7895	.8116
40	.6111	.7720	.6456	.6574	40	.7274	1.0599	.7922	.8145
50	.6134	.7766	.6484	.6603	50	.7294	1.0661	.7948	.8174
38°00′	.6157	.7813	.6511	.6632	**47°00′**	.7314	1.0724	.7975	.8203
10	.6180	.7860	.6539	.6661	10	.7333	1.0786	.8002	.8232
20	.6202	.7907	.6566	.6690	20	.7353	1.0850	.8028	.8261
30	.6225	.7954	.6594	.6720	30	.7373	1.0913	.8055	.8290
40	.6248	.8002	.6621	.6749	40	.7392	1.0977	.8082	.8319
50	.6271	.8050	.6649	.6778	50	.7412	1.1041	.8108	.8348
39°00′	.6293	.8098	.6676	.6807	**48°00′**	.7431	1.1106	.8135	.8378
10	.6316	.8146	.6704	.6836	10	.7451	1.1171	.8161	.8407
20	.6338	.8195	.6731	.6865	20	.7470	1.1237	.8188	.8436
30	.6361	.8243	.6758	.6894	30	.7490	1.1303	.8214	.8465
40	.6383	.8292	.6786	.6923	40	.7509	1.1369	.8241	.8494
50	.6406	.8342	.6813	.6952	50	.7528	1.1436	.8267	.8523
40°00′	.6428	.8391	.6840	.6981	**49°00′**	.7547	1.1504	.8294	.8552
10	.6450	.8441	.6868	.7010	10	.7566	1.1571	.8320	.8581
20	.6472	.8491	.6895	.7039	20	.7585	1.1640	.8347	.8610
30	.6494	.8541	.6922	.7069	30	.7604	1.1708	.8373	.8639
40	.6517	.8591	.6950	.7098	40	.7623	1.1778	.8400	.8668
50	.6539	.8642	.6977	.7127	50	.7642	1.1847	.8426	.8698
41°00′	.6561	.8693	.7004	.7156	**50°00′**	.7660	1.1918	.8452	.8727
10	.6583	.8744	.7031	.7185	10	.7679	1.1988	.8479	.8756
20	.6604	.8796	.7059	.7214	20	.7698	1.2059	.8505	.8785
30	.6626	.8847	.7086	.7243	30	.7716	1.2131	.8531	.8814
40	.6648	.8899	.7113	.7272	40	.7735	1.2203	.8558	.8843
50	.6670	.8952	.7140	.7301	50	.7753	1.2276	.8584	.8872
42°00′	.6691	.9004	.7167	.7330	**51°00′**	.7771	1.2349	.8610	.8901
10	.6713	.9057	.7195	.7359	10	.7790	1.2423	.8636	.8930
20	.6734	.9110	.7222	.7389	20	.7808	1.2497	.8663	.8959
30	.6756	.9163	.7249	.7418	30	.7826	1.2572	.8689	.8988
40	.6777	.9217	.7276	.7447	40	.7844	1.2647	.8715	.9018
50	.6799	.9271	.7303	.7476	50	.7862	1.2723	.8741	.9047
43°00′	.6820	.9325	.7330	.7505	**52°00′**	.7880	1.2799	.8767	.9076
10	.6841	.9380	.7357	.7534	10	.7898	1.2876	.8794	.9105
20	.6862	.9435	.7384	.7563	20	.7916	1.2954	.8820	.9134
30	.6884	.9490	.7411	.7592	30	.7934	1.3032	.8846	.9163
40	.6905	.9545	.7438	.7621	40	.7951	1.3111	.8872	.9192
50	.6926	.9601	.7465	.7650	50	.7969	1.3190	.8898	.9221
44°00′	.6947	.9657	.7492	.7679	**53°00′**	.7986	1.3270	.8924	.9250
10	.6967	.9713	.7519	.7709	10	.8004	1.3351	.8950	.9279
20	.6988	.9770	.7546	.7738	20	.8021	1.3432	.8976	.9308
30	.7009	.9827	.7573	.7767	30	.8039	1.3514	.9002	.9338
40	.7030	.9884	.7600	.7796	40	.8056	1.3597	.9028	.9367
50	.7050	.9942	.7627	.7825	50	.8073	1.3680	.9054	.9396
45°00′	.7071	1.0000	.7654	.7854	**54°00′**	.8090	1.3764	.9080	.9425

Angle	Sine	Tangent	Chord	Arc	Angle	Sine	Tangent	Chord	Arc
54°00′	.8090	1.3764	.9080	.9425	**63°00′**	.8910	1.9626	1.0450	1.0996
10	.8107	1.3848	.9106	.9454	10	.8923	1.9768	1.0475	1.1025
20	.8124	1.3934	.9132	.9483	20	.8936	1.9912	1.0500	1.1054
30	.8141	1.4019	.9157	.9512	30	.8949	2.0057	1.0524	1.1083
40	.8158	1.4106	.9183	.9541	40	.8962	2.0204	1.0549	1.1112
50	.8175	1.4193	.9209	.9570	50	.8975	2.0353	1.0574	1.1141
55°00′	.8192	1.4281	.9235	.9599	**64°00′**	.8988	2.0503	1.0598	1.1170
10	.8208	1.4370	.9261	.9628	10	.9001	2.0655	1.0623	1.1199
20	.8225	1.4460	.9287	.9657	20	.9013	2.0809	1.0648	1.1228
30	.8241	1.4550	.9312	.9687	30	.9026	2.0965	1.0672	1.1257
40	.8258	1.4641	.9338	.9716	40	.9038	2.1123	1.0697	1.1286
50	.8274	1.4733	.9364	.9745	50	.9051	2.1283	1.0721	1.1316
56°00′	.8290	1.4826	.9389	.9774	**65°00′**	.9063	2.1445	1.0746	1.1345
10	.8307	1.4919	.9415	.9803	10	.9075	2.1609	1.0771	1.1374
20	.8323	1.5013	.9441	.9832	20	.9088	2.1775	1.0795	1.1403
30	.8339	1.5108	.9466	.9861	30	.9100	2.1943	1.0820	1.1432
40	.8355	1.5204	.9492	.9890	40	.9112	2.2113	1.0844	1.1461
50	.8371	1.5301	.9518	.9919	50	.9124	2.2286	1.0868	1.1490
57°00′	.8387	1.5399	.9543	.9948	**66°00′**	.9135	2.2460	1.0893	1.1519
10	.8403	1.5497	.9569	.9977	10	.9147	2.2637	1.0917	1.1548
20	.8418	1.5597	.9594	1.0007	20	.9159	2.2817	1.0942	1.1577
30	.8434	1.5697	.9620	1.0036	30	.9171	2.2998	1.0966	1.1606
40	.8450	1.5798	.9645	1.0065	40	.9182	2.3183	1.0990	1.1636
50	.8465	1.5900	.9671	1.0094	50	.9194	2.3369	1.1014	1.1665
58°00′	.8480	1.6003	.9696	1.0123	**67°00′**	.9205	2.3559	1.1039	1.1694
10	.8496	1.6107	.9722	1.0152	10	.9216	2.3750	1.1063	1.1723
20	.8511	1.6212	.9747	1.0181	20	.9228	2.3945	1.1087	1.1752
30	.8526	1.6319	.9772	1.0210	30	.9239	2.4142	1.1111	1.1781
40	.8542	1.6426	.9798	1.0239	40	.9250	2.4342	1.1136	1.1810
50	.8557	1.6534	.9823	1.0268	50	.9261	2.4545	1.1160	1.1839
59°00′	.8572	1.6643	.9848	1.0297	**68°00′**	.9272	2.4751	1.1184	1.1868
10	.8587	1.6753	.9874	1.0327	10	.9283	2.4960	1.1208	1.1897
20	.8601	1.6864	.9899	1.0356	20	.9293	2.5172	1.1232	1.1926
30	.8616	1.6977	.9924	1.0385	30	.9304	2.5386	1.1256	1.1956
40	.8631	1.7090	.9950	1.0414	40	.9315	2.5605	1.1280	1.1985
50	.8646	1.7205	.9975	1.0443	50	.9325	2.5826	1.1304	1.2014
60°00′	.8660	1.7321	1.0000	1.0472	**69°00′**	.9336	2.6051	1.1328	1.2043
10	.8675	1.7437	1.0025	1.0501	10	.9346	2.6279	1.1352	1.2072
20	.8689	1.7556	1.0050	1.0530	20	.9356	2.6511	1.1376	1.2101
30	.8704	1.7675	1.0075	1.0559	30	.9367	2.6746	1.1400	1.2130
40	.8718	1.7796	1.0101	1.0588	40	.9377	2.6985	1.1424	1.2159
50	.8732	1.7917	1.0126	1.0617	50	.9387	2.7228	1.1448	1.2188
61°00′	.8746	1.8040	1.0151	1.0647	**70°00′**	.9397	2.7475	1.1472	1.2217
10	.8760	1.8165	1.0176	1.0676	10	.9407	2.7725	1.1495	1.2246
20	.8774	1.8291	1.0201	1.0705	20	.9417	2.7980	1.1519	1.2275
30	.8788	1.8418	1.0226	1.0734	30	.9426	2.8239	1.1543	1.2305
40	.8802	1.8546	1.0251	1.0763	40	.9436	2.8502	1.1567	1.2334
50	.8816	1.8676	1.0276	1.0792	50	.9446	2.8770	1.1590	1.2363
62°00′	.8829	1.8807	1.0301	1.0821	**71°00′**	.9455	2.9042	1.1614	1.2392
10	.8843	1.8940	1.0326	1.0850	10	.9465	2.9319	1.1638	1.2421
20	.8857	1.9074	1.0351	1.0879	20	.9474	2.9600	1.1661	1.2450
30	.8870	1.9210	1.0375	1.0908	30	.9483	2.9887	1.1685	1.2479
40	.8884	1.9347	1.0400	1.0937	40	.9492	3.0178	1.1709	1.2508
50	.8897	1.9486	1.0425	1.0966	50	.9502	3.0475	1.1732	1.2537
63°00′	.8910	1.9626	1.0450	1.0996	**72°00′**	.9511	3.0777	1.1756	1.2566

Angle	Sine	Tangent	Chord	Arc	Angle	Sine	Tangent	Chord	Arc
72°00′	.9511	3.0777	1.1756	1.2566	**81°00′**	.9877	6.3138	1.2989	1.4137
10	.9520	3.1084	1.1779	1.2595	10	.9881	6.4348	1.3011	1.4166
20	.9528	3.1397	1.1803	1.2625	20	.9886	6.5606	1.3033	1.4195
30	.9537	3.1716	1.1826	1.2654	30	.9890	6.6912	1.3055	1.4224
40	.9546	3.2041	1.1850	1.2683	40	.9894	6.8269	1.3077	1.4254
50	.9555	3.2371	1.1873	1.2712	50	.9899	6.9682	1.3099	1.4283
73°00′	.9563	3.2709	1.1896	1.2741	**82°00′**	.9903	7.1154	1.3121	1.4312
10	.9572	3.3052	1.1920	1.2770	10	.9907	7.2687	1.3143	1.4341
20	.9580	3.3402	1.1943	1.2799	20	.9911	7.4287	1.3165	1.4370
30	.9588	3.3759	1.1966	1.2828	30	.9914	7.5958	1.3187	1.4399
40	.9596	3.4124	1.1990	1.2857	40	.9918	7.7704	1.3209	1.4428
50	.9605	3.4495	1.2013	1.2886	50	.9922	7.9530	1.3231	1.4457
74°00′	.9613	3.4874	1.2036	1.2915	**83°00′**	.9925	8.1443	1.3252	1.4486
10	.9621	3.5261	1.2060	1.2945	10	.9929	8.3450	1.3274	1.4515
20	.9628	3.5656	1.2083	1.2974	20	.9932	8.5555	1.3296	1.4544
30	.9636	3.6059	1.2106	1.3003	30	.9936	8.7769	1.3318	1.4573
40	.9644	3.6470	1.2129	1.3032	40	.9939	9.0098	1.3339	1.4603
50	.9652	3.6891	1.2152	1.3061	50	.9942	9.2553	1.3361	1.4632
75°00′	.9659	3.7321	1.2175	1.3090	**84°00′**	.9945	9.5144	1.3383	1.4661
10	.9667	3.7760	1.2198	1.3119	10	.9948	9.7882	1.3404	1.4690
20	.9674	3.8208	1.2221	1.3148	20	.9951	10.078	1.3426	1.4719
30	.9681	3.8667	1.2244	1.3177	30	.9954	10.385	1.3447	1.4748
40	.9689	3.9136	1.2267	1.3206	40	.9957	10.712	1.3469	1.4777
50	.9696	3.9617	1.2290	1.3235	50	.9959	11.059	1.3490	1.4806
76°00′	.9703	4.0108	1.2313	1.3265	**85°00′**	.9962	11.430	1.3512	1.4835
10	.9710	4.0611	1.2336	1.3294	10	.9964	11.826	1.3533	1.4864
20	.9717	4.1126	1.2359	1.3323	20	.9967	12.251	1.3555	1.4893
30	.9724	4.1653	1.2382	1.3352	30	.9969	12.706	1.3576	1.4923
40	.9730	4.2193	1.2405	1.3381	40	.9971	13.197	1.3597	1.4952
50	.9737	4.2747	1.2428	1.3410	50	.9974	13.727	1.3619	1.4981
77°00′	.9744	4.3315	1.2450	1.3439	**86°00′**	.9976	14.301	1.3640	1.5010
10	.9750	4.3897	1.2473	1.3468	10	.9978	14.924	1.3661	1.5039
20	.9757	4.4494	1.2496	1.3497	20	.9980	15.605	1.3682	1.5068
30	.9763	4.5107	1.2518	1.3526	30	.9981	16.350	1.3704	1.5097
40	.9769	4.5736	1.2541	1.3555	40	.9983	17.169	1.3725	1.5126
50	.9775	4.6382	1.2564	1.3584	50	.9985	18.075	1.3746	1.5155
78°00′	.9781	4.7046	1.2586	1.3614	**87°00′**	.9986	19.081	1.3767	1.5184
10	.9787	4.7729	1.2609	1.3643	10	.9988	20.206	1.3788	1.5213
20	.9793	4.8430	1.2632	1.3672	20	.9989	21.470	1.3809	1.5243
30	.9799	4.9152	1.2654	1.3701	30	.9990	22.904	1.3830	1.5272
40	.9805	4.9894	1.2677	1.3730	40	.9992	24.542	1.3851	1.5301
50	.9811	5.0658	1.2699	1.3759	50	.9993	26.432	1.3872	1.5330
79°00′	.9816	5.1446	1.2722	1.3788	**88°00′**	.9994	28.636	1.3893	1.5359
10	.9822	5.2257	1.2744	1.3817	10	.9995	31.242	1.3914	1.5388
20	.9827	5.3093	1.2766	1.3846	20	.9996	34.368	1.3935	1.5417
30	.9833	5.3955	1.2789	1.3875	30	.9997	38.188	1.3956	1.5446
40	.9838	5.4845	1.2811	1.3904	40	.9997	42.964	1.3977	1.5475
50	.9843	5.5764	1.2833	1.3934	50	.9998	49.104	1.3997	1.5504
80°00′	.9848	5.6713	1.2856	1.3963	**89°00′**	.9998	57.290	1.4018	1.5533
10	.9853	5.7694	1.2878	1.3992	10	.9999	68.750	1.4039	1.5563
20	.9858	5.8708	1.2900	1.4021	20	.9999	85.940	1.4060	1.5592
30	.9863	5.9758	1.2922	1.4050	30	1.0000	114.59	1.4080	1.5621
40	.9868	6.0844	1.2945	1.4079	40	1.0000	171.89	1.4101	1.5650
50	.9872	6.1970	1.2967	1.4108	50	1.0000	343.77	1.4122	1.5679
81°00′	.9877	6.3138	1.2989	1.4137	**90°00′**	1.0000	——	1.4142	1.5708

TABLE II — POWERS AND ROOTS

EXPLANATION OF TABLE II

1. Squares and Cubes. The *squares* of numbers between 1.00 and 10.00 at intervals of .01 are given in column headed n^2. To find the square of any other number, divide (or multiply) the given number by 10 to reduce it to a number between 1 and 10; find the square of this last number; multiply (or divide) the square thus found by 10 twice as many times as you did the given number.

The *cube* is given in the column headed n^3. To find the cube of any number not between 1 and 10, first reduce that number to a number between 1 and 10 by dividing (or multiplying) by a power of 10. Multiply (or divide) the result found by three times as high a power of 10 as was used to reduce the given number.

2. Square Roots. The *square roots* of numbers between 1 and 10 are found in the column headed $\sqrt{n}$.

The square roots of numbers between 10 and 100 may be found in the column headed $\sqrt{10\,n}$.

The square roots of numbers between 100 and 1000 may be found in the column headed $\sqrt{n}$ by multiplying the given root by 10, since $\sqrt{100\,n} = 10\sqrt{n}$.

Other square roots may be found in a similar manner.

3. Cube Roots. The column headed:

$\sqrt[3]{n}$ gives cube roots of numbers between 1 and 10;
$\sqrt[3]{10\,n}$ gives cube roots of numbers between 10 and 100;
$\sqrt[3]{100\,n}$ gives cube roots of numbers between 100 and 1000.

To find the cube root of a number between 1000 and 10000, take 10 times the value found in the column headed $\sqrt[3]{n}$, since $\sqrt[3]{1000\,n} = 10\sqrt[3]{n}$.

Other cube roots may be found similarly.

n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$
1.00	1.0000	1.00000	3.16228	1.00000	1.00000	2.15443	4.64159
1.01	1.0201	1.00499	3.17805	1.03030	1.00332	2.16159	4.65701
1.02	1.0404	1.00995	3.19374	1.06121	1.00662	2.16870	4.67233
1.03	1.0609	1.01489	3.20936	1.09273	1.00990	2.17577	4.68755
1.04	1.0816	1.01980	3.22490	1.12486	1.01316	2.18279	4.70267
1.05	1.1025	1.02470	3.24037	1.15762	1.01640	2.18976	4.71769
1.06	1.1236	1.02956	3.25576	1.19102	1.01961	2.19669	4.73262
1.07	1.1449	1.03441	3.27109	1.22504	1.02281	2.20358	4.74746
1.08	1.1664	1.03923	3.28634	1.25971	1.02599	2.21042	4.76220
1.09	1.1881	1.04403	3.30151	1.29503	1.02914	2.21722	4.77686
1.10	1.2100	1.04881	3.31662	1.33100	1.03228	2.22398	4.79142
1.11	1.2321	1.05357	3.33167	1.36763	1.03540	2.23070	4.80590
1.12	1.2544	1.05830	3.34664	1.40493	1.03850	2.23738	4.82028
1.13	1.2769	1.06301	3.36155	1.44290	1.04158	2.24402	4.83459
1.14	1.2996	1.06771	3.37639	1.48154	1.04464	2.25062	4.84881
1.15	1.3225	1.07238	3.39116	1.52088	1.04769	2.25718	4.86294
1.16	1.3456	1.07703	3.40588	1.56090	1.05072	2.26370	4.87700
1.17	1.3689	1.08167	3.42053	1.60161	1.05373	2.27019	4.89097
1.18	1.3924	1.08628	3.43511	1.64303	1.05672	2.27664	4.90487
1.19	1.4161	1.09087	3.44964	1.68516	1.05970	2.28305	4.91868
1.20	1.4400	1.09545	3.46410	1.72800	1.06266	2.28943	4.93242
1.21	1.4641	1.10000	3.47851	1.77156	1.06560	2.29577	4.94609
1.22	1.4884	1.10454	3.49285	1.81585	1.06853	2.30208	4.95968
1.23	1.5129	1.10905	3.50714	1.86087	1.07144	2.30835	4.97319
1.24	1.5376	1.11355	3.52136	1.90662	1.07434	2.31459	4.98663
1.25	1.5625	1.11803	3.53553	1.95312	1.07722	2.32079	5.00000
1.26	1.5876	1.12250	3.54965	2.00038	1.08008	2.32697	5.01330
1.27	1.6129	1.12694	3.56371	2.04838	1.08293	2.33311	5.02653
1.28	1.6384	1.13137	3.57771	2.09715	1.08577	2.33921	5.03968
1.29	1.6641	1.13578	3.59166	2.14669	1.08859	2.34529	5.05277
1.30	1.6900	1.14018	3.60555	2.19700	1.09139	2.35133	5.06580
1.31	1.7161	1.14455	3.61939	2.24809	1.09418	2.35735	5.07875
1.32	1.7424	1.14891	3.63318	2.29997	1.09696	2.36333	5.09164
1.33	1.7689	1.15326	3.64692	2.35264	1.09972	2.36928	5.10447
1.34	1.7956	1.15758	3.66060	2.40610	1.10247	2.37521	5.11723
1.35	1.8225	1.16190	3.67423	2.46038	1.10521	2.38110	5.12993
1.36	1.8496	1.16619	3.68782	2.51546	1.10793	2.38697	5.14256
1.37	1.8769	1.17047	3.70135	2.57135	1.11064	2.39280	5.15514
1.38	1.9044	1.17473	3.71484	2.62807	1.11334	2.39861	5.16765
1.39	1.9321	1.17898	3.72827	2.68562	1.11602	2.40439	5.18010
1.40	1.9600	1.18322	3.74166	2.74400	1.11869	2.41014	5.19249
1.41	1.9881	1.18743	3.75500	2.80322	1.12135	2.41587	5.20483
1.42	2.0164	1.19164	3.76829	2.86329	1.12399	2.42156	5.21710
1.43	2.0449	1.19583	3.78153	2.92421	1.12662	2.42724	5.22932
1.44	2.0736	1.20000	3.79473	2.98598	1.12924	2.43288	5.24148
1.45	2.1025	1.20416	3.80789	3.04862	1.13185	2.43850	5.25359
1.46	2.1316	1.20830	3.82099	3.11214	1.13445	2.44409	5.26564
1.47	2.1609	1.21244	3.83406	3.17652	1.13703	2.44966	5.27763
1.48	2.1904	1.21655	3.84708	3.24179	1.13960	2.45520	5.28957
1.49	2.2201	1.22066	3.86005	3.30795	1.14216	2.46072	5.30146

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
1.50	2.2500	1.22474	3.87298	3.37500	1.14471	2.46621	5.31329
1.51	2.2801	1.22882	3.88587	3.44295	1.14725	2.47168	5.32507
1.52	2.3104	1.23288	3.89872	3.51181	1.14978	2.47712	5.33680
1.53	2.3409	1.23693	3.91152	3.58158	1.15230	2.48255	5.34848
1.54	2.3716	1.24097	3.92428	3.65226	1.15480	2.48794	5.36011
1.55	2.4025	1.24499	3.93700	3.72388	1.15729	2.49332	5.37169
1.56	2.4336	1.24900	3.94968	3.79642	1.15978	2.49867	5.38321
1.57	2.4649	1.25300	3.96232	3.86989	1.16225	2.50399	5.39469
1.58	2.4964	1.25698	3.97492	3.94431	1.16471	2.50930	5.40612
1.59	2.5281	1.26095	3.98748	4.01968	1.16717	2.51458	5.41750
1.60	2.5600	1.26491	4.00000	4.09600	1.16961	2.51984	5.42884
1.61	2.5921	1.26886	4.01248	4.17328	1.17204	2.52508	5.44012
1.62	2.6244	1.27279	4.02492	4.25153	1.17446	2.53030	5.45136
1.63	2.6569	1.27671	4.03733	4.33075	1.17687	2.53549	5.46256
1.64	2.6896	1.28062	4.04969	4.41094	1.17927	2.54067	5.47370
1.65	2.7225	1.28452	4.06202	4.49212	1.18167	2.54582	5.48481
1.66	2.7556	1.28841	4.07431	4.57430	1.18405	2.55095	5.49586
1.67	2.7889	1.29228	4.08656	4.65746	1.18642	2.55607	5.50688
1.68	2.8224	1.29615	4.09878	4.74163	1.18878	2.56116	5.51785
1.69	2.8561	1.30000	4.11096	4.82681	1.19114	2.56623	5.52877
1.70	2.8900	1.30384	4.12311	4.91300	1.19348	2.57128	5.53966
1.71	2.9241	1.30767	4.13521	5.00021	1.19582	2.57631	5.55050
1.72	2.9584	1.31149	4.14729	5.08845	1.19815	2.58133	5.56130
1.73	2.9929	1.31529	4.15933	5.17772	1.20046	2.58632	5.57205
1.74	3.0276	1.31909	4.17133	5.26802	1.20277	2.59129	5.58277
1.75	3.0625	1.32288	4.18330	5.35938	1.20507	2.59625	5.59344
1.76	3.0976	1.32665	4.19524	5.45178	1.20736	2.60118	5.60408
1.77	3.1329	1.33041	4.20714	5.54523	1.20964	2.60610	5.61467
1.78	3.1684	1.33417	4.21900	5.63975	1.21192	2.61100	5.62523
1.79	3.2041	1.33791	4.23084	5.73534	1.21418	2.61588	5.63574
1.80	3.2400	1.34164	4.24264	5.83200	1.21644	2.62074	5.64622
1.81	3.2761	1.34536	4.25441	5.92974	1.21869	2.62559	5.65665
1.82	3.3124	1.34907	4.26615	6.02857	1.22093	2.63041	5.66705
1.83	3.3489	1.35277	4.27785	6.12849	1.22316	2.63522	5.67741
1.84	3.3856	1.35647	4.28952	6.22950	1.22539	2.64001	5.68773
1.85	3.4225	1.36015	4.30116	6.33162	1.22760	2.64479	5.69802
1.86	3.4596	1.36382	4.31277	6.43486	1.22981	2.64954	5.70827
1.87	3.4969	1.36748	4.32435	6.53920	1.23201	2.65428	5.71848
1.88	3.5344	1.37113	4.33590	6.64467	1.23420	2.65901	5.72865
1.89	3.5721	1.37477	4.34741	6.75127	1.23639	2.66371	5.73879
1.90	3.6100	1.37840	4.35890	6.85900	1.23856	2.66840	5.74890
1.91	3.6481	1.38203	4.37035	6.96787	1.24073	2.67307	5.75897
1.92	3.6864	1.38564	4.38178	7.07789	1.24289	2.67773	5.76900
1.93	3.7249	1.38924	4.39318	7.18906	1.24505	2.68237	5.77900
1.94	3.7636	1.39284	4.40454	7.30138	1.24719	2.68700	5.78896
1.95	3.8025	1.39642	4.41588	7.41488	1.24933	2.69161	5.79889
1.96	3.8416	1.40000	4.42719	7.52954	1.25146	2.69620	5.80879
1.97	3.8809	1.40357	4.43847	7.64537	1.25359	2.70078	5.81865
1.98	3.9204	1.40712	4.44972	7.76239	1.25571	2.70534	5.82848
1.99	3.9601	1.41067	4.46094	7.88060	1.25782	2.70989	5.83827

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
2.00	4.0000	1.41421	4.47214	8.00000	1.25992	2.71442	5.84804
2.01	4.0401	1.41774	4.48330	8.12060	1.26202	2.71893	5.85777
2.02	4.0804	1.42127	4.49444	8.24241	1.26411	2.72344	5.86746
2.03	4.1209	1.42478	4.50555	8.36543	1.26619	2.72792	5.87713
2.04	4.1616	1.42829	4.51664	8.48966	1.26827	2.73239	5.88677
2.05	4.2025	1.43178	4.52769	8.61512	1.27033	2.73685	5.89637
2.06	4.2436	1.43527	4.53872	8.74182	1.27240	2.74129	5.90594
2.07	4.2849	1.43875	4.54973	8.86974	1.27445	2.74572	5.91548
2.08	4.3264	1.44222	4.56070	8.99891	1.27650	2.75014	5.92499
2.09	4.3681	1.44568	4.57165	9.12933	1.27854	2.75454	5.93447
2.10	4.4100	1.44914	4.58258	9.26100	1.28058	2.75892	5.94392
2.11	4.4521	1.45258	4.59347	9.39393	1.28261	2.76330	5.95334
2.12	4.4944	1.45602	4.60435	9.52813	1.28463	2.76766	5.96273
2.13	4.5369	1.45945	4.61519	9.66360	1.28665	2.77200	5.97209
2.14	4.5796	1.46287	4.62601	9.80034	1.28866	2.77633	5.98142
2.15	4.6225	1.46629	4.63681	9.93838	1.29066	2.78065	5.99073
2.16	4.6656	1.46969	4.64758	10.0777	1.29266	2.78495	6.00000
2.17	4.7089	1.47309	4.65833	10.2183	1.29465	2.78924	6.00925
2.18	4.7524	1.47648	4.66905	10.3602	1.29664	2.79352	6.01846
2.19	4.7961	1.47986	4.67974	10.5035	1.29862	2.79779	6.02765
2.20	4.8400	1.48324	4.69042	10.6480	1.30059	2.80204	6.03681
2.21	4.8841	1.48661	4.70106	10.7939	1.30256	2.80628	6.04594
2.22	4.9284	1.48997	4.71169	10.9410	1.30452	2.81050	6.05505
2.23	4.9729	1.49332	4.72229	11.0896	1.30648	2.81472	6.06413
2.24	5.0176	1.49666	4.73286	11.2394	1.30843	2.81892	6.07318
2.25	5.0625	1.50000	4.74342	11.3906	1.31037	2.82311	6.08220
2.26	5.1076	1.50333	4.75395	11.5432	1.31231	2.82728	6.09120
2.27	5.1529	1.50665	4.76445	11.6971	1.31424	2.83145	6.10017
2.28	5.1984	1.50997	4.77493	11.8524	1.31617	2.83560	6.10911
2.29	5.2441	1.51327	4.78539	12.0090	1.31809	2.83974	6.11803
2.30	5.2900	1.51658	4.79583	12.1670	1.32001	2.84387	6.12693
2.31	5.3361	1.51987	4.80625	12.3264	1.32192	2.84798	6.13579
2.32	5.3824	1.52315	4.81664	12.4872	1.32382	2.85209	6.14463
2.33	5.4289	1.52643	4.82701	12.6493	1.32572	2.85618	6.15345
2.34	5.4756	1.52971	4.83735	12.8129	1.32761	2.86026	6.16224
2.35	5.5225	1.53297	4.84768	12.9779	1.32950	2.86433	6.17101
2.36	5.5696	1.53623	4.85798	13.1443	1.33139	2.86838	6.17975
2.37	5.6169	1.53948	4.86826	13.3121	1.33326	2.87243	6.18846
2.38	5.6644	1.54272	4.87852	13.4813	1.33514	2.87646	6.19715
2.39	5.7121	1.54596	4.88876	13.6519	1.33700	2.88049	6.20582
2.40	5.7600	1.54919	4.89898	13.8240	1.33887	2.88450	6.21447
2.41	5.8081	1.55242	4.90918	13.9975	1.34072	2.88850	6.22308
2.42	5.8564	1.55563	4.91935	14.1725	1.34257	2.89249	6.23168
2.43	5.9049	1.55885	4.92950	14.3489	1.34442	2.89647	6.24025
2.44	5.9536	1.56205	4.93964	14.5268	1.34626	2.90044	6.24880
2.45	6.0025	1.56525	4.94975	14.7061	1.34810	2.90439	6.25732
2.46	6.0516	1.56844	4.95984	14.8869	1.34993	2.90834	6.26583
2.47	6.1009	1.57162	4.96991	15.0692	1.35176	2.91227	6.27431
2.48	6.1504	1.57480	4.97996	15.2530	1.35358	2.91620	6.28276
2.49	6.2001	1.57797	4.98999	15.4382	1.35540	2.92011	6.29119

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
2.50	6.2500	1.58114	5.00000	15.6250	1.35721	2.92402	6.29961
2.51	6.3001	1.58430	5.00999	15.8133	1.35902	2.92791	6.30799
2.52	6.3504	1.58745	5.01996	16.0030	1.36082	2.93179	6.31636
2.53	6.4009	1.59060	5.02991	16.1943	1.36262	2.93567	6.32470
2.54	6.4516	1.59374	5.03984	16.3871	1.36441	2.93953	6.33303
2.55	6.5025	1.59687	5.04975	16.5814	1.36620	2.94338	6.34133
2.56	6.5536	1.60000	5.05964	16.7772	1.36798	2.94723	6.34960
2.57	6.6049	1.60312	5.06952	16.9746	1.36976	2.95106	6.35786
2.58	6.6564	1.60624	5.07937	17.1735	1.37153	2.95488	6.36610
2.59	6.7081	1.60935	5.08920	17.3740	1.37330	2.95869	6.37431
2.60	6.7600	1.61245	5.09902	17.5760	1.37507	2.96250	6.38250
2.61	6.8121	1.61555	5.10882	17.7796	1.37683	2.96629	6.39068
2.62	6.8644	1.61864	5.11859	17.9847	1.37859	2.97007	6.39883
2.63	6.9169	1.62173	5.12835	18.1914	1.38034	2.97385	6.40696
2.64	6.9696	1.62481	5.13809	18.3997	1.38208	2.97761	6.41507
2.65	7.0225	1.62788	5.14782	18.6096	1.38383	2.98137	6.42316
2.66	7.0756	1.63095	5.15752	18.8211	1.38557	2.98511	6.43123
2.67	7.1289	1.63401	5.16720	19.0342	1.38730	2.98885	6.43928
2.68	7.1824	1.63707	5.17687	19.2488	1.38903	2.99257	6.44731
2.69	7.2361	1.64012	5.18652	19.4651	1.39076	2.99629	6.45531
2.70	7.2900	1.64317	5.19615	19.6830	1.39248	3.00000	6.46330
2.71	7.3441	1.64621	5.20577	19.9025	1.39419	3.00370	6.47127
2.72	7.3984	1.64924	5.21536	20.1236	1.39591	3.00739	6.47922
2.73	7.4529	1.65227	5.22494	20.3464	1.39761	3.01107	6.48715
2.74	7.5076	1.65529	5.23450	20.5708	1.39932	3.01474	6.49507
2.75	7.5625	1.65831	5.24404	20.7969	1.40102	3.01841	6.50296
2.76	7.6176	1.66132	5.25357	21.0246	1.40272	3.02206	6.51083
2.77	7.6729	1.66433	5.26308	21.2539	1.40441	3.02570	6.51868
2.78	7.7284	1.66733	5.27257	21.4850	1.40610	3.02934	6.52652
2.79	7.7841	1.67033	5.28205	21.7176	1.40778	3.03297	6.53434
2.80	7.8400	1.67332	5.29150	21.9520	1.40946	3.03659	6.54213
2.81	7.8961	1.67631	5.30094	22.1880	1.41114	3.04020	6.54991
2.82	7.9524	1.67929	5.31037	22.4258	1.41281	3.04380	6.55767
2.83	8.0089	1.68226	5.31977	22.6652	1.41448	3.04740	6.56541
2.84	8.0656	1.68523	5.32917	22.9063	1.41614	3.05098	6.57314
2.85	8.1225	1.68819	5.33854	23.1491	1.41780	3.05456	6.58084
2.86	8.1796	1.69115	5.34790	23.3937	1.41946	3.05813	6.58853
2.87	8.2369	1.69411	5.35724	23.6399	1.42111	3.06169	6.59620
2.88	8.2944	1.69706	5.36656	23.8879	1.42276	3.06524	6.60385
2.89	8.3521	1.70000	5.37587	24.1376	1.42440	3.06878	6.61149
2.90	8.4100	1.70294	5.38516	24.3890	1.42604	3.07232	6.61911
2.91	8.4681	1.70587	5.39444	24.6422	1.42768	3.07584	6.62671
2.92	8.5264	1.70880	5.40370	24.8971	1.42931	3.07936	6.63429
2.93	8.5849	1.71172	5.41295	25.1538	1.43094	3.08287	6.64185
2.94	8.6436	1.71464	5.42218	25.4122	1.43257	3.08638	6.64940
2.95	8.7025	1.71756	5.43139	25.6724	1.43419	3.08987	6.65693
2.96	8.7616	1.72047	5.44059	25.9343	1.43581	3.09336	6.66444
2.97	8.8209	1.72337	5.44977	26.1981	1.43743	3.09684	6.67194
2.98	8.8804	1.72627	5.45894	26.4636	1.43904	3.10031	6.67942
2.99	8.9401	1.72916	5.46809	26.7309	1.44065	3.10378	6.68688

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
3.00	9.0000	1.73205	5.47723	27.0000	1.44225	3.10723	6.69433
3.01	9.0601	1.73494	5.48635	27.2709	1.44385	3.11068	6.70176
3.02	9.1204	1.73781	5.49545	27.5436	1.44545	3.11412	6.70917
3.03	9.1809	1.74069	5.50454	27.8181	1.44704	3.11756	6.71657
3.04	9.2416	1.74356	5.51362	28.0945	1.44863	3.12098	6.72395
3.05	9.3025	1.74642	5.52268	28.3726	1.45022	3.12440	6.73132
3.06	9.3636	1.74929	5.53173	28.6526	1.45180	3.12781	6.73866
3.07	9.4249	1.75214	5.54076	28.9344	1.45338	3.13121	6.74600
3.08	9.4864	1.75499	5.54977	29.2181	1.45496	3.13461	6.75331
3.09	9.5481	1.75784	5.55878	29.5036	1.45653	3.13800	6.76061
3.10	9.6100	1.76068	5.56776	29.7910	1.45810	3.14138	6.76790
3.11	9.6721	1.76352	5.57674	30.0802	1.45967	3.14475	6.77517
3.12	9.7344	1.76635	5.58570	30.3713	1.46123	3.14812	6.78242
3.13	9.7969	1.76918	5.59464	30.6643	1.46279	3.15148	6.78966
3.14	9.8596	1.77200	5.60357	30.9591	1.46434	3.15483	6.79688
3.15	9.9225	1.77482	5.61249	31.2559	1.46590	3.15818	6.80409
3.16	9.9856	1.77764	5.62139	31.5545	1.46745	3.16152	6.81128
3.17	10.0489	1.78045	5.63028	31.8550	1.46899	3.16485	6.81846
3.18	10.1124	1.78326	5.63915	32.1574	1.47054	3.16817	6.82562
3.19	10.1761	1.78606	5.64801	32.4618	1.47208	3.17149	6.83277
3.20	10.2400	1.78885	5.65685	32.7680	1.47361	3.17480	6.83990
3.21	10.3041	1.79165	5.66569	33.0762	1.47515	3.17811	6.84702
3.22	10.3684	1.79444	5.67450	33.3862	1.47668	3.18140	6.85412
3.23	10.4329	1.79722	5.68331	33.6983	1.47820	3.18469	6.86121
3.24	10.4976	1.80000	5.69210	34.0122	1.47973	3.18798	6.86829
3.25	10.5625	1.80278	5.70088	34.3281	1.48125	3.19125	6.87534
3.26	10.6276	1.80555	5.70964	34.6460	1.48277	3.19452	6.88239
3.27	10.6929	1.80831	5.71839	34.9658	1.48428	3.19778	6.88942
3.28	10.7584	1.81108	5.72713	35.2876	1.48579	3.20104	6.89643
3.29	10.8241	1.81384	5.73585	35.6113	1.48730	3.20429	6.90344
3.30	10.8900	1.81659	5.74456	35.9370	1.48881	3.20753	6.91042
3.31	10.9561	1.81934	5.75326	36.2647	1.49031	3.21077	6.91740
3.32	11.0224	1.82209	5.76194	36.5944	1.49181	3.21400	6.92436
3.33	11.0889	1.82483	5.77062	36.9260	1.49330	3.21722	6.93130
3.34	11.1556	1.82757	5.77927	37.2597	1.49480	3.22044	6.93823
3.35	11.2225	1.83030	5.78792	37.5954	1.49629	3.22365	6.94515
3.36	11.2896	1.83303	5.79655	37.9331	1.49777	3.22686	6.95205
3.37	11.3569	1.83576	5.80517	38.2728	1.49926	3.23006	6.95894
3.38	11.4244	1.83848	5.81378	38.6145	1.50074	3.23325	6.96582
3.39	11.4921	1.84120	5.82237	38.9582	1.50222	3.23643	6.97268
3.40	11.5600	1.84391	5.83095	39.3040	1.50369	3.23961	6.97953
3.41	11.6281	1.84662	5.83952	39.6518	1.50517	3.24278	6.98637
3.42	11.6964	1.84932	5.84808	40.0017	1.50664	3.24595	6.99319
3.43	11.7649	1.85203	5.85662	40.3536	1.50810	3.24911	7.00000
3.44	11.8336	1.85472	5.86515	40.7076	1.50957	3.25227	7.00680
3.45	11.9025	1.85742	5.87367	41.0636	1.51103	3.25542	7.01358
3.46	11.9716	1.86011	5.88218	41.4217	1.51249	3.25856	7.02035
3.47	12.0409	1.86279	5.89067	41.7819	1.51394	3.26169	7.02711
3.48	12.1104	1.86548	5.89915	42.1442	1.51540	3.26482	7.03385
3.49	12.1801	1.86815	5.90762	42.5085	1.51685	3.26795	7.04058

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
3.50	12.2500	1.87083	5.91608	42.8750	1.51829	3.27107	7.04730
3.51	12.3201	1.87350	5.92453	43.2436	1.51974	3.27418	7.05400
3.52	12.3904	1.87617	5.93296	43.6142	1.52118	3.27729	7.06070
3.53	12.4609	1.87883	5.94138	43.9870	1.52262	3.28039	7.06738
3.54	12.5316	1.88149	5.94979	44.3619	1.52406	3.28348	7.07404
3.55	12.6025	1.88414	5.95819	44.7389	1.52549	3.28657	7.08070
3.56	12.6736	1.88680	5.96657	45.1180	1.52692	3.28965	7.08734
3.57	12.7449	1.88944	5.97495	45.4993	1.52835	3.29273	7.09397
3.58	12.8164	1.89209	5.98331	45.8827	1.52978	3.29580	7.10059
3.59	12.8881	1.89473	5.99166	46.2683	1.53120	3.29887	7.10719
3.60	12.9600	1.89737	6.00000	46.6560	1.53262	3.30193	7.11379
3.61	13.0321	1.90000	6.00833	47.0459	1.53404	3.30498	7.12037
3.62	13.1044	1.90263	6.01664	47.4379	1.53545	3.30803	7.12694
3.63	13.1769	1.90526	6.02495	47.8321	1.53686	3.31107	7.13349
3.64	13.2496	1.90788	6.03324	48.2285	1.53827	3.31411	7.14004
3.65	13.3225	1.91050	6.04152	48.6271	1.53968	3.31714	7.14657
3.66	13.3956	1.91311	6.04979	49.0279	1.54109	3.32017	7.15309
3.67	13.4689	1.91572	6.05805	49.4309	1.54249	3.32319	7.15960
3.68	13.5424	1.91833	6.06630	49.8360	1.54389	3.32621	7.16610
3.69	13.6161	1.92094	6.07454	50.2434	1.54529	3.32922	7.17258
3.70	13.6900	1.92354	6.08276	50.6530	1.54668	3.33222	7.17905
3.71	13.7641	1.92614	6.09098	51.0648	1.54807	3.33522	7.18552
3.72	13.8384	1.92873	6.09918	51.4788	1.54946	3.33822	7.19197
3.73	13.9129	1.93132	6.10737	51.8951	1.55085	3.34120	7.19840
3.74	13.9876	1.93391	6.11555	52.3136	1.55223	3.34419	7.20483
3.75	14.0625	1.93649	6.12372	52.7344	1.55362	3.34716	7.21125
3.76	14.1376	1.93907	6.13188	53.1574	1.55500	3.35014	7.21765
3.77	14.2129	1.94165	6.14003	53.5826	1.55637	3.35310	7.22405
3.78	14.2884	1.94422	6.14817	54.0102	1.55775	3.35607	7.23043
3.79	14.3641	1.94679	6.15630	54.4399	1.55912	3.35902	7.23680
3.80	14.4400	1.94936	6.16441	54.8720	1.56049	3.36198	7.24316
3.81	14.5161	1.95192	6.17252	55.3063	1.56186	3.36492	7.24950
3.82	14.5924	1.95448	6.18061	55.7430	1.56322	3.36786	7.25584
3.83	14.6689	1.95704	6.18870	56.1819	1.56459	3.37080	7.26217
3.84	14.7456	1.95959	6.19677	56.6231	1.56595	3.37373	7.26848
3.85	14.8225	1.96214	6.20484	57.0666	1.56731	3.37666	7.27479
3.86	14.8996	1.96469	6.21289	57.5125	1.56866	3.37958	7.28108
3.87	14.9769	1.96723	6.22093	57.9606	1.57001	3.38249	7.28736
3.88	15.0544	1.96977	6.22896	58.4111	1.57137	3.38540	7.29363
3.89	15.1321	1.97231	6.23699	58.8639	1.57271	3.38831	7.29989
3.90	15.2100	1.97484	6.24500	59.3190	1.57406	3.39121	7.30614
3.91	15.2881	1.97737	6.25300	59.7765	1.57541	3.39411	7.31238
3.92	15.3664	1.97990	6.26099	60.2363	1.57675	3.39700	7.31861
3.93	15.4449	1.98242	6.26897	60.6985	1.57809	3.39988	7.32483
3.94	15.5236	1.98494	6.27694	61.1630	1.57942	3.40277	7.33104
3.95	15.6025	1.98746	6.28490	61.6299	1.58076	3.40564	7.33723
3.96	15.6816	1.98997	6.29285	62.0991	1.58209	3.40851	7.34342
3.97	15.7609	1.99249	6.30079	62.5708	1.58342	3.41138	7.34960
3.98	15.8404	1.99499	6.30872	63.0448	1.58475	3.41424	7.35576
3.99	15.9201	1.99750	6.31664	63.5212	1.58608	3.41710	7.36192

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
4.00	16.0000	2.00000	6.32456	64.0000	1.58740	3.41995	7.36806
4.01	16.0801	2.00250	6.33246	64.4812	1.58872	3.42280	7.37420
4.02	16.1604	2.00499	6.34035	64.9648	1.59004	3.42564	7.38032
4.03	16.2409	2.00749	6.34823	65.4508	1.59136	3.42848	7.38644
4.04	16.3216	2.00998	6.35610	65.9393	1.59267	3.43131	7.39254
4.05	16.4025	2.01246	6.36396	66.4301	1.59399	3.43414	7.39864
4.06	16.4836	2.01494	6.37181	66.9234	1.59530	3.43697	7.40472
4.07	16.5649	2.01742	6.37966	67.4191	1.59661	3.43979	7.41080
4.08	16.6464	2.01990	6.38749	67.9173	1.59791	3.44260	7.41686
4.09	16.7281	2.02237	6.39531	68.4179	1.59922	3.44541	7.42291
4.10	16.8100	2.02485	6.40312	68.9210	1.60052	3.44822	7.42896
4.11	16.8921	2.02731	6.41093	69.4265	1.60182	3.45102	7.43499
4.12	16.9744	2.02978	6.41872	69.9345	1.60312	3.45382	7.44102
4.13	17.0569	2.03224	6.42651	70.4450	1.60441	3.45661	7.44703
4.14	17.1396	2.03470	6.43428	70.9579	1.60571	3.45939	7.45304
4.15	17.2225	2.03715	6.44205	71.4734	1.60700	3.46218	7.45904
4.16	17.3056	2.03961	6.44981	71.9913	1.60829	3.46496	7.46502
4.17	17.3889	2.04206	6.45755	72.5117	1.60958	3.46773	7.47100
4.18	17.4724	2.04450	6.46529	73.0346	1.61086	3.47050	7.47697
4.19	17.5561	2.04695	6.47302	73.5601	1.61215	3.47327	7.48292
4.20	17.6400	2.04939	6.48074	74.0880	1.61343	3.47603	7.48887
4.21	17.7241	2.05183	6.48845	74.6185	1.61471	3.47878	7.49481
4.22	17.8084	2.05426	6.49615	75.1514	1.61599	3.48154	7.50074
4.23	17.8929	2.05670	6.50384	75.6870	1.61726	3.48428	7.50666
4.24	17.9776	2.05913	6.51153	76.2250	1.61853	3.48703	7.51257
4.25	18.0625	2.06155	6.51920	76.7656	1.61981	3.48977	7.51847
4.26	18.1476	2.06398	6.52687	77.3088	1.62108	3.49250	7.52437
4.27	18.2329	2.06640	6.53452	77.8545	1.62234	3.49523	7.53025
4.28	18.3184	2.06882	6.54217	78.4028	1.62361	3.49796	7.53612
4.29	18.4041	2.07123	6.54981	78.9536	1.62487	3.50068	7.54199
4.30	18.4900	2.07364	6.55744	79.5070	1.62613	3.50340	7.54784
4.31	18.5761	2.07605	6.56506	80.0630	1.62739	3.50611	7.55369
4.32	18.6624	2.07846	6.57267	80.6216	1.62865	3.50882	7.55953
4.33	18.7489	2.08087	6.58027	81.1827	1.62991	3.51153	7.56535
4.34	18.8356	2.08327	6.58787	81.7465	1.63116	3.51423	7.57117
4.35	18.9225	2.08567	6.59545	82.3129	1.63241	3.51692	7.57698
4.36	19.0096	2.08806	6.60303	82.8819	1.63366	3.51962	7.58279
4.37	19.0969	2.09045	6.61060	83.4535	1.63491	3.52231	7.58858
4.38	19.1844	2.09284	6.61816	84.0277	1.63619	3.52499	7.59436
4.39	19.2721	2.09523	6.62571	84.6045	1.63740	3.52767	7.60014
4.40	19.3600	2.09762	6.63325	85.1840	1.63864	3.53035	7.60590
4.41	19.4481	2.10000	6.64078	85.7661	1.63988	3.53302	7.61166
4.42	19.5364	2.10238	6.64831	86.3509	1.64112	3.53569	7.61741
4.43	19.6249	2.10476	6.65582	86.9383	1.64236	3.53835	7.62315
4.44	19.7136	2.10713	6.66333	87.5284	1.64359	3.54101	7.62888
4.45	19.8025	2.10950	6.67083	88.1211	1.64483	3.54367	7.63461
4.46	19.8916	2.11187	6.67832	88.7165	1.64606	3.54632	7.64032
4.47	19.9809	2.11424	6.68581	89.3146	1.64729	3.54897	7.64603
4.48	20.0704	2.11660	6.69328	89.9154	1.64851	3.55162	7.65172
4.49	20.1601	2.11896	6.70075	90.5188	1.64974	3.55426	7.65741

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
4.50	20.2500	2.12132	6.70820	91.1250	1.65096	3.55689	7.66309
4.51	20.3401	2.12368	6.71565	91.7339	1.65219	3.55953	7.66877
4.52	20.4304	2.12603	6.72309	92.3454	1.65341	3.56215	7.67443
4.53	20.5209	2.12838	6.73053	92.9597	1.65462	3.56478	7.68009
4.54	20.6116	2.13073	6.73795	93.5767	1.65584	3.56740	7.68573
4.55	20.7025	2.13307	6.74537	94.1964	1.65706	3.57002	7.69137
4.56	20.7936	2.13542	6.75278	94.8188	1.65827	3.57263	7.69700
4.57	20.8849	2.13776	6.76018	95.4440	1.65948	3.57524	7.70262
4.58	20.9764	2.14009	6.76757	96.0719	1.66069	3.57785	7.70824
4.59	21.0681	2.14243	6.77495	96.7026	1.66190	3.58045	7.71384
4.60	21.1600	2.14476	6.78233	97.3360	1.66310	3.58305	7.71944
4.61	21.2521	2.14709	6.78970	97.9722	1.66431	3.58564	7.72503
4.62	21.3444	2.14942	6.79706	98.6111	1.66551	3.58823	7.73061
4.63	21.4369	2.15174	6.80441	99.2528	1.66671	3.59082	7.73619
4.64	21.5296	2.15407	6.81175	99.8973	1.66791	3.59340	7.74175
4.65	21.6225	2.15639	6.81909	100.545	1.66911	3.59598	7.74731
4.66	21.7156	2.15870	6.82642	101.195	1.67030	3.59856	7.75286
4.67	21.8089	2.16102	6.83374	101.848	1.67150	3.60113	7.75840
4.68	21.9024	2.16333	6.84105	102.503	1.67269	3.60370	7.76394
4.69	21.9961	2.16564	6.84836	103.162	1.67388	3.60626	7.76946
4.70	22.0900	2.16795	6.85565	103.823	1.67507	3.60883	7.77498
4.71	22.1841	2.17025	6.86294	104.487	1.67626	3.61138	7.78049
4.72	22.2784	2.17256	6.87023	105.154	1.67744	3.61394	7.78599
4.73	22.3729	2.17486	6.87750	105.824	1.67863	3.61649	7.79149
4.74	22.4676	2.17715	6.88477	106.496	1.67981	3.61903	7.79697
4.75	22.5625	2.17945	6.89202	107.172	1.68099	3.62158	7.80245
4.76	22.6576	2.18174	6.89928	107.850	1.68217	3.62412	7.80793
4.77	22.7529	2.18403	6.90652	108.531	1.68334	3.62665	7.81339
4.78	22.8484	2.18632	6.91375	109.215	1.68452	3.62919	7.81885
4.79	22.9441	2.18861	6.92098	109.902	1.68569	3.63172	7.82429
4.80	23.0400	2.19089	6.92820	110.592	1.68687	3.63424	7.82974
4.81	23.1361	2.19317	6.93542	111.285	1.68804	3.63676	7.83517
4.82	23.2324	2.19545	6.94262	111.980	1.68920	3.63928	7.84059
4.83	23.3289	2.19773	6.94982	112.679	1.69037	3.64180	7.84601
4.84	23.4256	2.20000	6.95701	113.380	1.69154	3.64431	7.85142
4.85	23.5225	2.20227	6.96419	114.084	1.69270	3.64682	7.85683
4.86	23.6196	2.20454	6.97137	114.791	1.69386	3.64932	7.86222
4.87	23.7169	2.20681	6.97854	115.501	1.69503	3.65182	7.86761
4.88	23.8144	2.20907	6.98570	116.214	1.69619	3.65432	7.87299
4.89	23.9121	2.21133	6.99285	116.930	1.69734	3.65681	7.87837
4.90	24.0100	2.21359	7.00000	117.649	1.69850	3.65931	7.88374
4.91	24.1081	2.21585	7.00714	118.371	1.69965	3.66179	7.88909
4.92	24.2064	2.21811	7.01427	119.095	1.70081	3.66428	7.89445
4.93	24.3049	2.22036	7.02140	119.823	1.70196	3.66676	7.89979
4.94	24.4036	2.22261	7.02851	120.554	1.70311	3.66924	7.90513
4.95	24.5025	2.22486	7.03562	121.287	1.70426	3.67171	7.91046
4.96	24.6016	2.22711	7.04273	122.024	1.70540	3.67418	7.91578
4.97	24.7009	2.22935	7.04982	122.763	1.70655	3.67665	7.92110
4.98	24.8004	2.23159	7.05691	123.506	1.70769	3.67911	7.92641
4.99	24.9001	2.23383	7.06399	124.251	1.70884	3.68157	7.93171

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
5.00	25.0000	2.23607	7.07107	125.000	1.70998	3.68403	7.93701
5.01	25.1001	2.23830	7.07814	125.752	1.71112	3.68649	7.94229
5.02	25.2004	2.24054	7.08520	126.506	1.71225	3.68894	7.94757
5.03	25.3009	2.24277	7.09225	127.264	1.71339	3.69138	7.95285
5.04	25.4016	2.24499	7.09930	128.024	1.71452	3.69383	7.95811
5.05	25.5025	2.24722	7.10634	128.788	1.71566	3.69627	7.96337
5.06	25.6036	2.24944	7.11337	129.554	1.71679	3.69871	7.96863
5.07	25.7049	2.25167	7.12039	130.324	1.71792	3.70114	7.97387
5.08	25.8064	2.25389	7.12741	131.097	1.71905	3.70357	7.97911
5.09	25.9081	2.25610	7.13442	131.872	1.72017	3.70600	7.98434
5.10	26.0100	2.25832	7.14143	132.651	1.72130	3.70843	7.98957
5.11	26.1121	2.26053	7.14843	133.433	1.72242	3.71085	7.99479
5.12	26.2144	2.26274	7.15542	134.218	1.72355	3.71327	8.00000
5.13	26.3169	2.26495	7.16240	135.006	1.72467	3.71569	8.00520
5.14	26.4196	2.26716	7.16938	135.797	1.72579	3.71810	8.01040
5.15	26.5225	2.26936	7.17635	136.591	1.72691	3.72051	8.01559
5.16	26.6256	2.27156	7.18331	137.388	1.72802	3.72292	8.02078
5.17	26.7289	2.27376	7.19027	138.188	1.72914	3.72532	8.02596
5.18	26.8324	2.27596	7.19722	138.992	1.73025	3.72772	8.03113
5.19	26.9361	2.27816	7.20417	139.798	1.73137	3.73012	8.03629
5.20	27.0400	2.28035	7.21110	140.608	1.73248	3.73251	8.04145
5.21	27.1441	2.28254	7.21803	141.421	1.73359	3.73490	8.04660
5.22	27.2484	2.28473	7.22496	142.237	1.73470	3.73729	8.05175
5.23	27.3529	2.28692	7.23187	143.056	1.73580	3.73968	8.05689
5.24	27.4576	2.28910	7.23878	143.878	1.73691	3.74206	8.06202
5.25	27.5625	2.29129	7.24569	144.703	1.73801	3.74443	8.06714
5.26	27.6676	2.29347	7.25259	145.532	1.73912	3.74681	8.07226
5.27	27.7729	2.29565	7.25948	146.363	1.74022	3.74918	8.07737
5.28	27.8784	2.29783	7.26636	147.198	1.74132	3.75155	8.08248
5.29	27.9841	2.30000	7.27324	148.036	1.74242	3.75392	8.08758
5.30	28.0900	2.30217	7.28011	148.877	1.74351	3.75629	8.09267
5.31	28.1961	2.30434	7.28697	149.721	1.74461	3.75865	8.09776
5.32	28.3024	2.30651	7.29383	150.569	1.74570	3.76101	8.10284
5.33	28.4089	2.30868	7.30068	151.419	1.74680	3.76336	8.10791
5.34	28.5156	2.31084	7.30753	152.273	1.74789	3.76571	8.11298
5.35	28.6225	2.31301	7.31437	153.130	1.74898	3.76806	8.11804
5.36	28.7296	2.31517	7.32120	153.991	1.75007	3.77041	8.12310
5.37	28.8369	2.31733	7.32803	154.854	1.75116	3.77275	8.12814
5.38	28.9444	2.31948	7.33485	155.721	1.75224	3.77509	8.13319
5.39	29.0521	2.32164	7.34166	156.591	1.75333	3.77743	8.13822
5.40	29.1600	2.32379	7.34847	157.464	1.75441	3.77976	8.14325
5.41	29.2681	2.32594	7.35527	158.340	1.75549	3.78209	8.14828
5.42	29.3764	2.32809	7.36206	159.220	1.75657	3.78442	8.15329
5.43	29.4849	2.33024	7.36885	160.103	1.75765	3.78675	8.15831
5.44	29.5936	2.33238	7.37564	160.989	1.75873	3.78907	8.16331
5.45	29.7025	2.33452	7.38241	161.879	1.75981	3.79139	8.16831
5.46	29.8116	2.33666	7.38918	162.771	1.76088	3.79371	8.17330
5.47	29.9209	2.33880	7.39594	163.667	1.76196	3.79603	8.17829
5.48	30.0304	2.34094	7.40270	164.567	1.76303	3.79834	8.18327
5.49	30.1401	2.34307	7.40945	165.469	1.76410	3.80065	8.18824

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
5.50	30.2500	2.34521	7.41620	166.375	1.76517	3.80295	8.19321
5.51	30.3601	2.34734	7.42294	167.284	1.76624	3.80526	8.19818
5.52	30.4704	2.34947	7.42967	168.197	1.76731	3.80756	8.20313
5.53	30.5809	2.35160	7.43640	169.112	1.76838	3.80985	8.20808
5.54	30.6916	2.35372	7.44312	170.031	1.76944	3.81215	8.21303
5.55	30.8025	2.35584	7.44983	170.954	1.77051	3.81444	8.21797
5.56	30.9136	2.35797	7.45654	171.880	1.77157	3.81673	8.22290
5.57	31.0249	2.36008	7.46324	172.809	1.77263	3.81902	8.22783
5.58	31.1364	2.36220	7.46994	173.741	1.77369	3.82130	8.23275
5.59	31.2481	2.36432	7.47663	174.677	1.77475	3.82358	8.23766
5.60	31.3600	2.36643	7.48331	175.616	1.77581	3.82586	8.24257
5.61	31.4721	2.36854	7.48999	176.558	1.77686	3.82814	8.24747
5.62	31.5844	2.37065	7.49667	177.504	1.77792	3.83041	8.25237
5.63	31.6969	2.37276	7.50333	178.454	1.77897	3.83268	8.25726
5.64	31.8096	2.37487	7.50999	179.406	1.78003	3.83495	8.26215
5.65	31.9225	2.37697	7.51665	180.362	1.78108	3.83722	8.26703
5.66	32.0356	2.37908	7.52330	181.321	1.78213	3.83948	8.27190
5.67	32.1489	2.38118	7.52994	182.284	1.78318	3.84174	8.27677
5.68	32.2624	2.38328	7.53658	183.250	1.78422	3.84399	8.28164
5.69	32.3761	2.38537	7.54321	184.220	1.78527	3.84625	8.28649
5.70	32.4900	2.38747	7.54983	185.193	1.78632	3.84850	8.29134
5.71	32.6041	2.38956	7.55645	186.169	1.78736	3.85075	8.29619
5.72	32.7184	2.39165	7.56307	187.149	1.78840	3.85300	8.30103
5.73	32.8329	2.39374	7.56968	188.133	1.78944	3.85524	8.30587
5.74	32.9476	2.39583	7.57628	189.119	1.79048	3.85748	8.31069
5.75	33.0625	2.39792	7.58288	190.109	1.79152	3.85972	8.31552
5.76	33.1776	2.40000	7.58947	191.103	1.79256	3.86196	8.32034
5.77	33.2929	2.40208	7.59605	192.100	1.79360	3.86419	8.32515
5.78	33.4084	2.40416	7.60263	193.101	1.79463	3.86642	8.32995
5.79	33.5241	2.40624	7.60920	194.105	1.79567	3.86865	8.33476
5.80	33.6400	2.40832	7.61577	195.112	1.79670	3.87088	8.33955
5.81	33.7561	2.41039	7.62234	196.123	1.79773	3.87310	8.34434
5.82	33.8724	2.41247	7.62889	197.137	1.79876	3.87532	8.34913
5.83	33.9889	2.41454	7.63544	198.155	1.79979	3.87754	8.35390
5.84	34.1056	2.41661	7.64199	199.177	1.80082	3.87975	8.35868
5.85	34.2225	2.41868	7.64853	200.202	1.80185	3.88197	8.36345
5.86	34.3396	2.42074	7.65506	201.230	1.80288	3.88418	8.36821
5.87	34.4569	2.42281	7.66159	202.262	1.80390	3.88639	8.37297
5.88	34.5744	2.42487	7.66812	203.297	1.80492	3.88859	8.37772
5.89	34.6921	2.42693	7.67463	204.336	1.80595	3.89080	8.38247
5.90	34.8100	2.42899	7.68115	205.379	1.80697	3.89300	8.38721
5.91	34.9281	2.43105	7.68765	206.425	1.80799	3.89519	8.39194
5.92	35.0464	2.43311	7.69415	207.475	1.80901	3.89739	8.39667
5.93	35.1649	2.43516	7.70065	208.528	1.81003	3.89958	8.40140
5.94	35.2836	2.43721	7.70714	209.585	1.81104	3.90177	8.40612
5.95	35.4025	2.43926	7.71362	210.645	1.81206	3.90396	8.41083
5.96	35.5216	2.44131	7.72010	211.709	1.81307	3.90615	8.41554
5.97	35.6409	2.44336	7.72658	212.776	1.81409	3.90833	8.42025
5.98	35.7604	2.44540	7.73305	213.847	1.81510	3.91051	8.42494
5.99	35.8801	2.44745	7.73951	214.922	1.81611	3.91269	8.42964

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
6.00	36.0000	2.44949	7.74597	216.000	1.81712	3.91487	8.43433
6.01	36.1201	2.45153	7.75242	217.082	1.81813	3.91704	8.43901
6.02	36.2404	2.45357	7.75887	218.167	1.81914	3.91921	8.44369
6.03	36.3609	2.45561	7.76531	219.256	1.82014	3.92138	8.44836
6.04	36.4816	2.45764	7.77174	220.349	1.82115	3.92355	8.45303
6.05	36.6025	2.45967	7.77817	221.445	1.82215	3.92571	8.45769
6.06	36.7236	2.46171	7.78460	222.545	1.82316	3.92787	8.46235
6.07	36.8449	2.46374	7.79102	223.649	1.82416	3.93003	8.46700
6.08	36.9664	2.46577	7.79744	224.756	1.82516	3.93219	8.47165
6.09	37.0881	2.46779	7.80385	225.867	1.82616	3.93434	8.47629
6.10	37.2100	2.46982	7.81025	226.981	1.82716	3.93650	8.48093
6.11	37.3321	2.47184	7.81665	228.099	1.82816	3.93865	8.48556
6.12	37.4544	2.47386	7.82304	229.221	1.82915	3.94079	8.49018
6.13	37.5769	2.47588	7.82943	230.346	1.83015	3.94294	8.49481
6.14	37.6996	2.47790	7.83582	231.476	1.83115	3.94508	8.49942
6.15	37.8225	2.47992	7.84219	232.608	1.83214	3.94722	8.50403
6.16	37.9456	2.48193	7.84857	233.745	1.83313	3.94936	8.50864
6.17	38.0689	2.48395	7.85493	234.885	1.83412	3.95150	8.51324
6.18	38.1924	2.48596	7.86130	236.029	1.83511	3.95363	8.51784
6.19	38.3161	2.48797	7.86766	237.177	1.83610	3.95576	8.52243
6.20	38.4400	2.48998	7.87401	238.328	1.83709	3.95789	8.52702
6.21	38.5641	2.49199	7.88036	239.483	1.83808	3.96002	8.53160
6.22	38.6884	2.49399	7.88670	240.642	1.83906	3.96214	8.53618
6.23	38.8129	2.49600	7.89303	241.804	1.84005	3.96427	8.54075
6.24	38.9376	2.49800	7.89937	242.971	1.84103	3.96638	8.54532
6.25	39.0625	2.50000	7.90569	244.141	1.84202	3.96850	8.54988
6.26	39.1876	2.50200	7.91202	245.314	1.84300	3.97062	8.55444
6.27	39.3129	2.50400	7.91833	246.492	1.84398	3.97273	8.55899
6.28	39.4384	2.50599	7.92465	247.673	1.84496	3.97484	8.56354
6.29	39.5641	2.50799	7.93095	248.858	1.84594	3.97695	8.56808
6.30	39.6900	2.50998	7.93725	250.047	1.84691	3.97906	8.57262
6.31	39.8161	2.51197	7.94355	251.240	1.84789	3.98116	8.57715
6.32	39.9424	2.51396	7.94984	252.436	1.84887	3.98326	8.58168
6.33	40.0689	2.51595	7.95613	253.636	1.84984	3.98536	8.58620
6.34	40.1956	2.51794	7.96241	254.840	1.85082	3.98746	8.59072
6.35	40.3225	2.51992	7.96869	256.048	1.85179	3.98956	8.59524
6.36	40.4496	2.52190	7.97496	257.259	1.85276	3.99165	8.59975
6.37	40.5769	2.52389	7.98123	258.475	1.85373	3.99374	8.60425
6.38	40.7044	2.52587	7.98749	259.694	1.85470	3.99583	8.60875
6.39	40.8321	2.52784	7.99375	260.917	1.85567	3.99792	8.61325
6.40	40.9600	2.52982	8.00000	262.144	1.85664	4.00000	8.61774
6.41	41.0881	2.53180	8.00625	263.375	1.85760	4.00208	8.62222
6.42	41.2164	2.53377	8.01249	264.609	1.85857	4.00416	8.62671
6.43	41.3449	2.53574	8.01873	265.848	1.85953	4.00624	8.63118
6.44	41.4736	2.53772	8.02496	267.090	1.86050	4.00832	8.63566
6.45	41.6025	2.53969	8.03119	268.336	1.86146	4.01039	8.64012
6.46	41.7316	2.54165	8.03741	269.586	1.86242	4.01246	8.64459
6.47	41.8609	2.54362	8.04363	270.840	1.86338	4.01453	8.64904
6.48	41.9904	2.54558	8.04984	272.098	1.86434	4.01660	8.65350
6.49	42.1201	2.54755	8.05605	273.359	1.86530	4.01866	8.65795

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
6.50	42.2500	2.54951	8.06226	274.625	1.86626	4.02073	8.66239
6.51	42.3801	2.55147	8.06846	275.894	1.86721	4.02279	8.66683
6.52	42.5104	2.55343	8.07465	277.168	1.86817	4.02485	8.67127
6.53	42.6409	2.55539	8.08084	278.445	1.86912	4.02690	8.67570
6.54	42.7716	2.55734	8.08703	279.726	1.87008	4.02896	8.68012
6.55	42.9025	2.55930	8.09321	281.011	1.87103	4.03101	8.68455
6.56	43.0336	2.56125	8.09938	282.300	1.87198	4.03306	8.68896
6.57	43.1649	2.56320	8.10555	283.593	1.87293	4.03511	8.69338
6.58	43.2964	2.56515	8.11172	284.890	1.87388	4.03715	8.69778
6.59	43.4281	2.56710	8.11788	286.191	1.87483	4.03920	8.70219
6.60	43.5600	2.56905	8.12404	287.496	1.87578	4.04124	8.70659
6.61	43.6921	2.57099	8.13019	288.805	1.87672	4.04328	8.71098
6.62	43.8244	2.57294	8.13634	290.118	1.87767	4.04532	8.71537
6.63	43.9569	2.57488	8.14248	291.434	1.87862	4.04735	8.71976
6.64	44.0896	2.57682	8.14862	292.755	1.87956	4.04939	8.72414
6.65	44.2225	2.57876	8.15475	294.080	1.88050	4.05142	8.72852
6.66	44.3556	2.58070	8.16088	295.408	1.88144	4.05345	8.73289
6.67	44.4889	2.58263	8.16701	296.741	1.88239	4.05548	8.73726
6.68	44.6224	2.58457	8.17313	298.078	1.88333	4.05750	8.74162
6.69	44.7561	2.58650	8.17924	299.418	1.88427	4.05953	8.74598
6.70	44.8900	2.58844	8.18535	300.763	1.88520	4.06155	8.75034
6.71	45.0241	2.59037	8.19146	302.112	1.88614	4.06357	8.75469
6.72	45.1584	2.59230	8.19756	303.464	1.88708	4.06559	8.75904
6.73	45.2929	2.59422	8.20366	304.821	1.88801	4.06760	8.76338
6.74	45.4276	2.59615	8.20975	306.182	1.88895	4.06961	8.76772
6.75	45.5625	2.59808	8.21584	307.547	1.88988	4.07163	8.77205
6.76	45.6976	2.60000	8.22192	308.916	1.89081	4.07364	8.77638
6.77	45.8329	2.60192	8.22800	310.289	1.89175	4.07564	8.78071
6.78	45.9684	2.60384	8.23408	311.666	1.89268	4.07765	8.78503
6.79	46.1041	2.60576	8.24015	313.047	1.89361	4.07965	8.78935
6 80	46.2400	2.60768	8.24621	314.432	1.89454	4.08166	8.79366
6.81	46.3761	2.60960	8.25227	315.821	1.89546	4.08365	8.79797
6.82	46.5124	2.61151	8.25833	317.215	1.89639	4.08565	8.80227
6.83	46.6489	2.61343	8.26438	318.612	1.89732	4.08765	8.80657
6.84	46.7856	2.61534	8.27043	320.014	1.89824	4.08964	8.81087
6.85	46.9225	2.61725	8.27647	321.419	1.89917	4.09163	8.81516
6.86	47.0596	2.61916	8.28251	322.829	1.90009	4.09362	8.81945
6.87	47.1969	2.62107	8.28855	324.243	1.90102	4.09561	8.82373
6.88	47.3344	2.62298	8.29458	325.661	1.90194	4.09760	8.82801
6.89	47.4721	2.62488	8.30060	327.083	1.90286	4.09958	8.83228
6.90	47.6100	2.62679	8.30662	328.509	1.90378	4.10157	8.83656
6.91	47.7481	2.62869	8.31264	329.939	1.90470	4.10355	8.84082
6.92	47.8864	2.63059	8.31865	331.374	1.90562	4.10552	8.84509
6.93	48.0249	2.63249	8.32466	332.813	1.90653	4.10750	8.84934
6.94	48.1636	2.63439	8.33067	334.255	1.90745	4.10948	8.85360
6.95	48.3025	2.63629	8.33667	335.702	1.90837	4.11145	8.85785
6.96	48.4416	2.63818	8.34266	337.154	1.90928	4.11342	8.86210
6.97	48.5809	2.64008	8.34865	338.609	1.91019	4.11539	8.86634
6.98	48.7204	2.64197	8.35464	340.068	1.91111	4.11736	8.87058
6.99	48.8601	2.64386	8.36062	341.532	1.91202	4.11932	8.87481

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
7.00	49.0000	2.64575	8.36660	343.000	1.91293	4.12129	8.87904
7.01	49.1401	2.64764	8.37257	344.472	1.91384	4.12325	8.88327
7.02	49.2804	2.64953	8.37854	345.948	1.91475	4.12521	8.88749
7.03	49.4209	2.65141	8.38451	347.429	1.91566	4.12716	8.89171
7.04	49.5616	2.65330	8.39047	348.914	1.91657	4.12912	8.89592
7.05	49.7025	2.65518	8.39643	350.403	1.91747	4.13107	8.90013
7.06	49.8436	2.65707	8.40238	351.896	1.91838	4.13303	8.90434
7.07	49.9849	2.65895	8.40833	353.393	1.91929	4.13498	8.90854
7.08	50.1264	2.66083	8.41427	354.895	1.92019	4.13693	8.91274
7.09	50.2681	2.66271	8.42021	356.401	1.92109	4.13887	8.91693
7.10	50.4100	2.66458	8.42615	357.911	1.92200	4.14082	8.92112
7.11	50.5521	2.66646	8.43208	359.425	1.92290	4.14276	8.92531
7.12	50.6944	2.66833	8.43801	360.944	1.92380	4.14470	8.92949
7.13	50.8369	2.67021	8.44393	362.467	1.92470	4.14664	8.93367
7.14	50.9796	2.67208	8.44985	363.994	1.92560	4.14858	8.93784
7.15	51.1225	2.67395	8.45577	365.526	1.92650	4.15052	8.94201
7.16	51.2656	2.67582	8.46168	367.062	1.92740	4.15245	8.94618
7.17	51.4089	2.67769	8.46759	368.602	1.92829	4.15438	8.95034
7.18	51.5524	2.67955	8.47349	370.146	1.92919	4.15631	8.95450
7.19	51.6961	2.68142	8.47939	371.695	1.93008	4.15824	8.95866
7.20	51.8400	2.68328	8.48528	373.248	1.93098	4.16017	8.96281
7.21	51.9841	2.68514	8.49117	374.805	1.93187	4.16209	8.96696
7.22	52.1284	2.68701	8.49706	376.367	1.93277	4.16402	8.97110
7.23	52.2729	2.68887	8.50294	377.933	1.93366	4.16594	8.97524
7.24	52.4176	2.69072	8.50882	379.503	1.93455	4.16786	8.97938
7.25	52.5625	2.69258	8.51469	381.078	1.93544	4.16978	8.98351
7.26	52.7076	2.69444	8.52056	382.657	1.93633	4.17169	8.98764
7.27	52.8529	2.69629	8.52643	384.241	1.93722	4.17361	8.99176
7.28	52.9984	2.69815	8.53229	385.828	1.93810	4.17552	8.99588
7.29	53.1441	2.70000	8.53815	387.420	1.93899	4.17743	9.00000
7.30	53.2900	2.70185	8.54400	389.017	1.93988	4.17934	9.00411
7.31	53.4361	2.70370	8.54985	390.618	1.94076	4.18125	9.00822
7.32	53.5824	2.70555	8.55570	392.223	1.94165	4.18315	9.01233
7.33	53.7289	2.70740	8.56154	393.833	1.94253	4.18506	9.01643
7.34	53.8756	2.70924	8.56738	395.447	1.94341	4.18696	9.02053
7.35	54.0225	2.71109	8.57321	397.065	1.94430	4.18886	9.02462
7.36	54.1696	2.71293	8.57904	398.688	1.94518	4.19076	9.02871
7.37	54.3169	2.71477	8.58487	400.316	1.94606	4.19266	9.03280
7.38	54.4644	2.71662	8.59069	401.947	1.94694	4.19455	9.03689
7.39	54.6121	2.71846	8.59651	403.583	1.94782	4.19644	9.04097
7.40	54.7600	2.72029	8.60233	405.224	1.94870	4.19834	9.04504
7.41	54.9081	2.72213	8.60814	406.869	1.94957	4.20023	9.04911
7.42	55.0564	2.72397	8.61394	408.518	1.95045	4.20212	9.05318
7.43	55.2049	2.72580	8.61974	410.172	1.95132	4.20400	9.05725
7.44	55.3536	2.72764	8.62554	411.831	1.95220	4.20589	9.06131
7.45	55.5025	2.72947	8.63134	413.494	1.95307	4.20777	9.06537
7.46	55.6516	2.73130	8.63713	415.161	1.95395	4.20965	9.06942
7.47	55.8009	2.73313	8.64292	416.833	1.95482	4.21153	9.07347
7.48	55.9504	2.73496	8.64870	418.509	1.95569	4.21341	9.07752
7.49	56.1001	2.73679	8.65448	420.190	1.95656	4.21529	9.08156

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
7.50	56.2500	2.73861	8.66025	421.875	1.95743	4.21716	9.08560
7.51	56.4001	2.74044	8.66603	423.565	1.95830	4.21904	9.08964
7.52	56.5504	2.74226	8.67179	425.259	1.95917	4.22091	9.09367
7.53	56.7009	2.74408	8.67756	426.958	1.96004	4.22278	9.09770
7.54	56.8516	2.74591	8.68332	428.661	1.96091	4.22465	9.10173
7.55	57.0025	2.74773	8.68907	430.369	1.96177	4.22651	9.10575
7.56	57.1536	2.74955	8.69483	432.081	1.96264	4.22838	9.10977
7.57	57.3049	2.75136	8.70057	433.798	1.96350	4.23024	9.11378
7.58	57.4564	2.75318	8.70632	435.520	1.96437	4.23210	9.11779
7.59	57.6081	2.75500	8.71206	437.245	1.96523	4.23396	9.12180
7.60	57.7600	2.75681	8.71780	438.976	1.96610	4.23582	9.12581
7.61	57.9121	2.75862	8.72353	440.711	1.96696	4.23768	9.12981
7.62	58.0644	2.76043	8.72926	442.451	1.96782	4.23954	9.13380
7.63	58.2169	2.76225	8.73499	444.195	1.96868	4.24139	9.13780
7.64	58.3696	2.76405	8.74071	445.944	1.96954	4.24324	9.14179
7.65	58.5225	2.76586	8.74643	447.697	1.97040	4.24509	9.14577
7.66	58.6756	2.76767	8.75214	449.455	1.97126	4.24694	9.14976
7.67	58.8289	2.76948	8.75785	451.218	1.97211	4.24879	9.15374
7.68	58.9824	2.77128	8.76356	452.985	1.97297	4.25063	9.15771
7.69	59.1361	2.77308	8.76926	454.757	1.97383	4.25248	9.16169
7.70	59.2900	2.77489	8.77496	456.533	1.97468	4.25432	9.16566
7.71	59.4441	2.77669	8.78066	458.314	1.97554	4.25616	9.16962
7.72	59.5984	2.77849	8.78635	460.100	1.97639	4.25800	9.17359
7.73	59.7529	2.78029	8.79204	461.890	1.97724	4.25984	9.17754
7.74	59.9076	2.78209	8.79773	463.685	1.97809	4.26167	9.18150
7.75	60.0625	2.78388	8.80341	465.484	1.97895	4.26351	9.18545
7.76	60.2176	2.78568	8.80909	467.289	1.97980	4.26534	9.18940
7.77	60.3729	2.78747	8.81476	469.097	1.98065	4.26717	9.19335
7.78	60.5284	2.78927	8.82043	470.911	1.98150	4.26900	9.19729
7.79	60.6841	2.79106	8.82610	472.729	1.98234	4.27083	9.20123
7.80	60.8400	2.79285	8.83176	474.552	1.98319	4.27266	9.20516
7.81	60.9961	2.79464	8.83742	476.380	1.98404	4.27448	9.20910
7.82	61.1524	2.79643	8.84308	478.212	1.98489	4.27631	9.21302
7.83	61.3089	2.79821	8.84873	480.049	1.98573	4.27813	9.21695
7.84	61.4656	2.80000	8.85438	481.890	1.98658	4.27995	9.22087
7.85	61.6225	2.80179	8.86002	483.737	1.98742	4.28177	9.22479
7.86	61.7796	2.80357	8.86566	485.588	1.98826	4.28359	9.22871
7.87	61.9369	2.80535	8.87130	487.443	1.98911	4.28540	9.23262
7.88	62.0944	2.80713	8.87694	489.304	1.98995	4.28722	9.23653
7.89	62.2521	2.80891	8.88257	491.169	1.99079	4.28903	9.24043
7.90	62.4100	2.81069	8.88819	493.039	1.99163	4.29084	9.24434
7.91	62.5681	2.81247	8.89382	494.914	1.99247	4.29265	9.24823
7.92	62.7264	2.81425	8.89944	496.793	1.99331	4.29446	9.25213
7.93	62.8849	2.81603	8.90505	498.677	1.99415	4.29627	9.25602
7.94	63.0436	2.81780	8.91067	500.566	1.99499	4.29807	9.25991
7.95	63.2025	2.81957	8.91628	502.460	1.99582	4.29987	9.26380
7.96	63.3616	2.82135	8.92188	504.358	1.99666	4.30168	9.26768
7.97	63.5209	2.82312	8.92749	506.262	1.99750	4.30348	9.27156
7.98	63.6804	2.82489	8.93308	508.170	1.99833	4.30528	9.27544
7.99	63.8401	2.82666	8.93868	510.082	1.99917	4.30707	9.27931

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
8.00	64.0000	2.82843	8.94427	512.000	2.00000	4.30887	9.28318
8.01	64.1601	2.83019	8.94986	513.922	2.00083	4.31066	9.28704
8.02	64.3204	2.83196	8.95545	515.850	2.00167	4.31246	9.29091
8.03	64.4809	2.83373	8.96103	517.782	2.00250	4.31425	9.29477
8.04	64.6416	2.83549	8.96660	519.718	2.00333	4.31604	9.29862
8.05	64.8025	2.83725	8.97218	521.660	2.00416	4.31783	9.30248
8.06	64.9636	2.83901	8.97775	523.607	2.00499	4.31961	9.30633
8.07	65.1249	2.84077	8.98332	525.558	2.00582	4.32140	9.31018
8.08	65.2864	2.84253	8.98888	527.514	2.00664	4.32318	9.31402
8.09	65.4481	2.84429	8.99444	529.475	2.00747	4.32497	9.31786
8.10	65.6100	2.84605	9.00000	531.441	2.00830	4.32675	9.32170
8.11	65.7721	2.84781	9.00555	533.412	2.00912	4.32853	9.32553
8.12	65.9344	2.84956	9.01110	535.387	2.00995	4.33031	9.32936
8.13	66.0969	2.85132	9.01665	537.368	2.01078	4.33208	9.33319
8.14	66.2596	2.85307	9.02219	539.353	2.01160	4.33386	9.33702
8.15	66.4225	2.85482	9.02774	541.343	2.01242	4.33563	9.34084
8.16	66.5856	2.85657	9.03327	543.338	2.01325	4.33741	9.34466
8.17	66.7489	2.85832	9.03881	545.339	2.01407	4.33918	9.34847
8.18	66.9124	2.86007	9.04434	547.343	2.01489	4.34095	9.35229
8.19	67.0761	2.86182	9.04986	549.353	2.01571	4.34271	9.35610
8.20	67.2400	2.86356	9.05539	551.368	2.01653	4.34448	9.35990
8.21	67.4041	2.86531	9.06091	553.388	2.01735	4.34625	9.36370
8.22	67.5684	2.86705	9.06642	555.412	2.01817	4.34801	9.36751
8.23	67.7329	2.86880	9.07193	557.442	2.01899	4.34977	9.37130
8.24	67.8976	2.87054	9.07744	559.476	2.01980	4.35153	9.37510
8.25	68.0625	2.87228	9.08295	561.516	2.02062	4.35329	9.37889
8.26	68.2276	2.87402	9.08845	563.560	2.02144	4.35505	9.38268
8.27	68.3929	2.87576	9.09395	565.609	2.02225	4.35681	9.38646
8.28	68.5584	2.87750	9.09945	567.664	2.02307	4.35856	9.39024
8.29	68.7241	2.87924	9.10494	569.723	2.02388	4.36032	9.39402
8.30	68.8900	2.88097	9.11043	571.787	2.02469	4.36207	9.39780
8.31	69.0561	2.88271	9.11592	573.856	2.02551	4.36382	9.40157
8.32	69.2224	2.88444	9.12140	575.930	2.02632	4.36557	9.40534
8.33	69.3889	2.88617	9.12688	578.010	2.02713	4.36732	9.40911
8.34	69.5556	2.88791	9.13236	580.094	2.02794	4.36907	9.41287
8.35	69.7225	2.88964	9.13783	582.183	2.02875	4.37081	9.41663
8.36	69.8896	2.89137	9.14330	584.277	2.02956	4.37256	9.42039
8.37	70.0569	2.89310	9.14877	586.376	2.03037	4.37430	9.42414
8.38	70.2244	2.89482	9.15423	588.480	2.03118	4.37604	9.42789
8.39	70.3921	2.89655	9.15969	590.590	2.03199	4.37778	9.43164
8.40	70.5600	2.89828	9.16515	592.704	2.03279	4.37952	9.43539
8.41	70.7281	2.90000	9.17061	594.823	2.03360	4.38126	9.43913
8.42	70.8964	2.90172	9.17606	596.948	2.03440	4.38299	9.44287
8.43	71.0649	2.90345	9.18150	599.077	2.03521	4.38473	9.44661
8.44	71.2336	2.90517	9.18695	601.212	2.03601	4.38646	9.45034
8.45	71.4025	2.90689	9.19239	603.351	2.03682	4.38819	9.45407
8.46	71.5716	2.90861	9.19783	605.496	2.03762	4.38992	9.45780
8.47	71.7409	2.91033	9.20326	607.645	2.03842	4.39165	9.46152
8.48	71.9104	2.91204	9.20869	609.800	2.03923	4.39338	9.46525
8.49	72.0801	2.91376	9.21412	611.960	2.04003	4.39510	9.46897

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
8.50	72.2500	2.91548	9.21954	614.125	2.04083	4.39683	9.47268
8.51	72.4201	2.91719	9.22497	616.295	2.04163	4.39855	9.47640
8.52	72.5904	2.91890	9.23038	618.470	2.04243	4.40028	9.48011
8.53	72.7609	2.92062	9.23580	620.650	2.04323	4.40200	9.48381
8.54	72.9316	2.92233	9.24121	622.836	2.04402	4.40372	9.48752
8.55	73.1025	2.92404	9.24662	625.026	2.04482	4.40543	9.49122
8.56	73.2736	2.92575	9.25203	627.222	2.04562	4.40715	9.49492
8.57	73.4449	2.92746	9.25743	629.423	2.04641	4.40887	9.49861
8.58	73.6164	2.92916	9.26283	631.629	2.04721	4.41058	9.50231
8.59	73.7881	2.93087	9.26823	633.840	2.04801	4.41229	9.50600
8.60	73.9600	2.93258	9.27362	636.056	2.04880	4.41400	9.50969
8.61	74.1321	2.93428	9.27901	638.277	2.04959	4.41571	9.51337
8.62	74.3044	2.93598	9.28440	640.504	2.05039	4.41742	9.51705
8.63	74.4769	2.93769	9.28978	642.736	2.05118	4.41913	9.52073
8.64	74.6496	2.93939	9.29516	644.973	2.05197	4.42084	9.52441
8.65	74.8225	2.94109	9.30054	647.215	2.05276	4.42254	9.52808
8.66	74.9956	2.94279	9.30591	649.462	2.05355	4.42425	9.53175
8.67	75.1689	2.94449	9.31128	651.714	2.05434	4.42595	9.53542
8.68	75.3424	2.94618	9.31665	653.972	2.05513	4.42765	9.53908
8.69	75.5161	2.94788	9.32202	656.235	2.05592	4.42935	9.54274
8.70	75.6900	2.94958	9.32738	658.503	2.05671	4.43105	9.54640
8.71	75.8641	2.95127	9.33274	660.776	2.05750	4.43274	9.55006
8.72	76.0384	2.95296	9.33809	663.055	2.05828	4.43444	9.55371
8.73	76.2129	2.95466	9.34345	665.339	2.05907	4.43613	9.55736
8.74	76.3876	2.95635	9.34880	667.628	2.05986	4.43783	9.56101
8.75	76.5625	2.95804	9.35414	669.922	2.06064	4.43952	9.56466
8.76	76.7376	2.95973	9.35949	672.221	2.06143	4.44121	9.56830
8.77	76.9129	2.96142	9.36483	674.526	2.06221	4.44290	9.57194
8.78	77.0884	2.96311	9.37017	676.836	2.06299	4.44459	9.57557
8.79	77.2641	2.96479	9.37550	679.151	2.06378	4.44627	9.57921
8.80	77.4400	2.96648	9.38083	681.472	2.06456	4.44796	9.58284
8.81	77.6161	2.96816	9.38616	683.798	2.06534	4.44964	9.58647
8.82	77.7924	2.96985	9.39149	686.129	2.06612	4.45133	9.59009
8.83	77.9689	2.97153	9.39681	688.465	2.06690	4.45301	9.59372
8.84	78.1456	2.97321	9.40213	690.807	2.06768	4.45469	9.59734
8.85	78.3225	2.97489	9.40744	693.154	2.06846	4.45637	9.60095
8.86	78.4996	2.97658	9.41276	695.506	2.06924	4.45805	9.60457
8.87	78.6769	2.97825	9.41807	697.864	2.07002	4.45972	9.60818
8.88	78.8544	2.97993	9.42338	700.227	2.07080	4.46140	9.61179
8.89	79.0321	2.98161	9.42868	702.595	2.07157	4.46307	9.61540
8.90	79.2100	2.98329	9.43398	704.969	2.07235	4.46475	9.61900
8.91	79.3881	2.98496	9.43928	707.348	2.07313	4.46642	9.62260
8.92	79.5664	2.98664	9.44458	709.732	2.07390	4.46809	9.62620
8.93	79.7449	2.98831	9.44987	712.122	2.07468	4.46976	9.62980
8.94	79.9236	2.98998	9.45516	714.517	2.07545	4.47142	9.63339
8.95	80.1025	2.99166	9.46044	716.917	2.07622	4.47309	9.63698
8.96	80.2816	2.99333	9.46573	719.323	2.07700	4.47476	9.64057
8.97	80.4609	2.99500	9.47101	721.734	2.07777	4.47642	9.64415
8.98	80.6404	2.99666	9.47629	724.151	2.07854	4.47808	9.64774
8.99	80.8201	2.99833	9.48156	726.573	2.07931	4.47974	9.65132

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
9.00	81.0000	3.00000	9.48683	729.000	2.08008	4.48140	9.65489
9.01	81.1801	3.00167	9.49210	731.433	2.08085	4.48306	9.65847
9.02	81.3604	3.00333	9.49737	733.871	2.08162	4.48472	9.66204
9.03	81.5409	3.00500	9.50263	736.314	2.08239	4.48638	9.66561
9.04	81.7216	3.00666	9.50789	738.763	2.08316	4.48803	9.66918
9.05	81.9025	3.00832	9.51315	741.218	2.08393	4.48969	9.67274
9.06	82.0836	3.00998	9.51840	743.677	2.08470	4.49134	9.67630
9.07	82.2649	3.01164	9.52365	746.143	2.08546	4.49299	9.67986
9.08	82.4464	3.01330	9.52890	748.613	2.08623	4.49464	9.68342
9.09	82.6281	3.01496	9.53415	751.089	2.08699	4.49629	9.68697
9.10	82.8100	3.01662	9.53939	753.571	2.08776	4.49794	9.69052
9.11	82.9921	3.01828	9.54463	756.058	2.08852	4.49959	9.69407
9.12	83.1744	3.01993	9.54987	758.551	2.08929	4.50123	9.69762
9.13	83.3569	3.02159	9.55510	761.048	2.09005	4.50288	9.70116
9.14	83.5396	3.02324	9.56033	763.552	2.09081	4.50452	9.70470
9.15	83.7225	3.02490	9.56556	766.061	2.09158	4.50616	9.70824
9.16	83.9056	3.02655	9.57079	768.575	2.09234	4.50781	9.71177
9.17	84.0889	3.02820	9.57601	771.095	2.09310	4.50945	9.71531
9.18	84.2724	3.02985	9.58123	773.621	2.09386	4.51108	9.71884
9.19	84.4561	3.03150	9.58645	776.152	2.09462	4.51272	9.72236
9.20	84.6400	3.03315	9.59166	778.688	2.09538	4.51436	9.72589
9.21	84.8241	3.03480	9.59687	781.230	2.09614	4.51599	9.72941
9.22	85.0084	3.03645	9.60208	783.777	2.09690	4.51763	9.73293
9.23	85.1929	3.03809	9.60729	786.330	2.09765	4.51926	9.73645
9.24	85.3776	3.03974	9.61249	788.889	2.09841	4.52089	9.73996
9.25	85.5625	3.04138	9.61769	791.453	2.09917	4.52252	9.74348
9.26	85.7476	3.04302	9.62289	794.023	2.09992	4.52415	9.74699
9.27	85.9329	3.04467	9.62808	796.598	2.10068	4.52578	9.75049
9.28	86.1184	3.04631	9.63328	799.179	2.10144	4.52740	9.75400
9.29	86.3041	3.04795	9.63846	801.765	2.10219	4.52903	9.75750
9.30	86.4900	3.04959	9.64365	804.357	2.10294	4.53065	9.76100
9.31	86.6761	3.05123	9.64883	806.954	2.10370	4.53228	9.76450
9.32	86.8624	3.05287	9.65401	809.558	2.10445	4.53390	9.76799
9.33	87.0489	3.05450	9.65919	812.166	2.10520	4.53552	9.77148
9.34	87.2356	3.05614	9.66437	814.781	2.10595	4.53714	9.77497
9.35	87.4225	3.05778	9.66954	817.400	2.10671	4.53876	9.77846
9.36	87.6096	3.05941	9.67471	820.026	2.10746	4.54038	9.78195
9.37	87.7969	3.06105	9.67988	822.657	2.10821	4.54199	9.78543
9.38	87.9844	3.06268	9.68504	825.294	2.10896	4.54361	9.78891
9.39	88.1721	3.06431	9.69020	827.936	2.10971	4.54522	9.79239
9.40	88.3600	3.06594	9.69536	830.584	2.11045	4.54684	9.79586
9.41	88.5481	3.06757	9.70052	833.238	2.11120	4.54845	9.79933
9.42	88.7364	3.06920	9.70567	835.897	2.11195	4.55006	9.80280
9.43	88.9249	3.07083	9.71082	838.562	2.11270	4.55167	9.80627
9.44	89.1136	3.07246	9.71597	841.232	2.11344	4.55328	9.80974
9.45	89.3025	3.07409	9.72111	843.909	2.11419	4.55488	9.81320
9.46	89.4916	3.07571	9.72625	846.591	2.11494	4.55649	9.81666
9.47	89.6809	3.07734	9.73139	849.278	2.11568	4.55809	9.82012
9.48	89.8704	3.07896	9.73653	851.971	2.11642	4.55970	9.82357
9.49	90.0601	3.08058	9.74166	854.670	2.11717	4.56130	9.82703

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
9.50	90.2500	3.08221	9.74679	857.375	2.11791	4.56290	9.83048
9.51	90.4401	3.08383	9.75192	860.085	2.11865	4.56450	9.83392
9.52	90.6304	3.08545	9.75705	862.801	2.11940	4.56610	9.83737
9.53	90.8209	3.08707	9.76217	865.523	2.12014	4.56770	9.84081
9.54	91.0116	3.08869	9.76729	868.251	2.12088	4.56930	9.84425
9.55	91.2025	3.09031	9.77241	870.984	2.12162	4.57089	9.84769
9.56	91.3936	3.09192	9.77753	873.723	2.12236	4.57249	9.85113
9.57	91.5849	3.09354	9.78264	876.467	2.12310	4.57408	9.85456
9.58	91.7764	3.09516	9.78775	879.218	2.12384	4.57567	9.85799
9.59	91.9681	3.09677	9.79285	881.974	2.12458	4.57727	9.86142
9.60	92.1600	3.09839	9.79796	884.736	2.12532	4.57886	9.86485
9.61	92.3521	3.10000	9.80306	887.504	2.12605	4.58045	9.86827
9.62	92.5444	3.10161	9.80816	890.277	2.12679	4.58204	9.87169
9.63	92.7369	3.10322	9.81326	893.056	2.12753	4.58362	9.87511
9.64	92.9296	3.10483	9.81835	895.841	2.12826	4.58521	9.87853
9.65	93.1225	3.10644	9.82344	898.632	2.12900	4.58679	9.88195
9.66	93.3156	3.10805	9.82853	901.429	2.12974	4.58838	9.88536
9.67	93.5089	3.10966	9.83362	904.231	2.13047	4.58996	9.88877
9.68	93.7024	3.11127	9.83870	907.039	2.13120	4.59154	9.89217
9.69	93.8961	3.11288	9.84378	909.853	2.13194	4.59312	9.89558
9.70	94.0900	3.11448	9.84886	912.673	2.13267	4.59470	9.89898
9.71	94.2841	3.11609	9.85393	915.499	2.13340	4.59628	9.90238
9.72	94.4784	3.11769	9.85901	918.330	2.13414	4.59786	9.90578
9.73	94.6729	3.11929	9.86408	921.167	2.13487	4.59943	9.90918
9.74	94.8676	3.12090	9.86914	924.010	2.13560	4.60101	9.91257
9.75	95.0625	3.12250	9.87421	926.859	2.13633	4.60258	9.91596
9.76	95.2576	3.12410	9.87927	929.714	2.13706	4.60416	9.91935
9.77	95.4529	3.12570	9.88433	932.575	2.13779	4.60573	9.92274
9.78	95.6484	3.12730	9.88939	935.441	2.13852	4.60730	9.92612
9.79	95.8441	3.12890	9.89444	938.314	2.13925	4.60887	9.92950
9.80	96.0400	3.13050	9.89949	941.192	2.13997	4.61044	9.93288
9.81	96.2361	3.13209	9.90454	944.076	2.14070	4.61200	9.93626
9.82	96.4324	3.13369	9.90959	946.966	2.14143	4.61357	9.93964
9.83	96.6289	3.13528	9.91464	949.862	2.14216	4.61514	9.94301
9.84	96.8256	3.13688	9.91968	952.764	2.14288	4.61670	9.94638
9.85	97.0225	3.13847	9.92472	955.672	2.14361	4.61826	9.94975
9.86	97.2196	3.14006	9.92975	958.585	2.14433	4.61983	9.95311
9.87	97.4169	3.14166	9.93479	961.505	2.14506	4.62139	9.95648
9.88	97.6144	3.14325	9.93982	964.430	2.14578	4.62295	9.95984
9.89	97.8121	3.14484	9.94485	967.362	2.14651	4.62451	9.96320
9.90	98.0100	3.14643	9.94987	970.299	2.14723	4.62607	9.96655
9.91	98.2081	3.14802	9.95490	973.242	2.14795	4.62762	9.96991
9.92	98.4064	3.14960	9.95992	976.191	2.14867	4.62918	9.97326
9.93	98.6049	3.15119	9.96494	979.147	2.14940	4.63073	9.97661
9.94	98.8036	3.15278	9.96995	982.108	2.15012	4.63229	9.97996
9.95	99.0025	3.15436	9.97497	985.075	2.15084	4.63384	9.98331
9.96	99.2016	3.15595	9.97998	988.048	2.15156	4.63539	9.98665
9.97	99.4009	3.15753	9.98499	991.027	2.15228	4.63694	9.98999
9.98	99.6004	3.15911	9.98999	994.012	2.15300	4.63849	9.99333
9.99	99.8001	3.16070	9.99500	997.003	2.15372	4.64004	9.99667

TABLE III — IMPORTANT NUMBERS

A. *Units of Length*

English Units

12 inches (in.) = 1 foot (ft.)
3 feet = 1 yard (yd.)
$5\frac{1}{2}$ yards = 1 rod (rd.)
320 rods = 1 mile (mi.)

Metric Units

10 millimeters (mm.) = 1 centimeter (cm.)
10 centimeters = 1 decimeter (dm.)
10 decimeters = 1 meter (m.)
10 meters = 1 dekameter (Dm.)
1000 meters = 1 kilometer (Km.)

English to Metric

1 in. = 2.5400 cm.
1 ft. = 30.480 cm.
1 mi. = 1.6093 Km.

Metric to English

1 cm. = 0.3937 in.
1 m. = 39.37 in. = 3.2808 ft.
1 Km. = 0.6214 mi.

B. *Units of Area or Surface*

1 square yard = 9 square feet = 1296 square inches
1 acre (A.) = 160 square rods = 4840 square yards
1 square mile = 640 acres = 102400 square rods

C. *Units of Measurement of Capacity*

Dry Measure

2 pints (pt.) = 1 quart (qt.)
8 quarts = 1 peck (pk.)
4 pecks = 1 bushel (bu.)

Liquid Measure

4 gills (gi.) = 1 pint (pt.)
2 pints = 1 quart (qt.)
4 quarts = 1 gallon (gal.)
1 gallon = 231 cu. in.

D. *Metric Units to English Units*

1 liter = 1000 cu. cm. = 61.02 cu. in. = 1.0567 liquid quarts
1 quart = .94636 liter = 946.36 cu. cm.
1000 grams = 1 kilogram (Kg.) = 2.2046 pounds (lb.)
1 pound = .453593 kilogram = 453.59 grams

E. *Other Numbers*

π = ratio of circumference to diameter of a circle
= 3.14159265
1 radian = angle subtended by an arc equal to the radius
$= 57^\circ\, 17'\, 44''.8 = 57^\circ.2957795 = 180^\circ/\pi$
1 degree = 0.01745329 radian, or $\pi/180$ radians
Weight of 1 cu. ft. of water = 62.425 lb.

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Plane and solid geometry