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as presented on 3/6/2014.
Date

We recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science with a major in Mathematics

Advisor — Department Mathematical Sciences — Date 3/19/2014

Member — Department Mathematical Sciences — Date 3/18/14

Member — Department Mathematical Sciences — Date 3/18/14

Outside Member — Affiliation Physics & Eng. — Date 3/20/14

APPROVED

Program Director — Department Mathematical Sciences — Date 3/20/2014

Dean — College Math, Nat. Sci & Tech. — Date 03/20/2014

Dean, School of Graduate Studies and Research — Saundra F DeLauder — Date 3/27/2014

OPTICAL SOLITONS IN MULTI-DIMENSIONS WITH SPATIO-TEMPORAL DISPERSION AND NON-KERR LAW NONLINEARITY

by

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A THESIS

Submitted in partial fulfillment of the requirements for the
degree of Master of Science in
the Mathematical Physics Graduate Program
of Delaware State University

DOVER, DELAWARE
March 2014

TITLE

OPTICAL SOLITONS IN MULTI-DIMENSIONS WITH SPATIO-TEMPORAL DISPERSION AND NON-KERR LAW NONLINEARITY

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DEDICATION

I dedicate my dissertation work to my family and teachers. A special feeling of gratitude to my loving parents, whose words of encouragement and push for tenacity ring in my ears. They have never left my side and are very special.

I also dedicate this dissertation to my teachers who have supported me throughout the process. I will always appreciate all they have done.

ACKNOWLEDGEMENTS

I would like to thank Dr. Biswas, my supervisor, for his many suggestions and constant support during this research.

I should also mention that my graduate studies in USA were supported in part by the Research Group of Dr. Liu.

I am grateful to my parents for their patience and love. Without them this work would never have come into existence.

ABSTRACT

This thesis studies the dynamics of optical solitons in multi-dimensions with spatio-temporal dispersion and non-Kerr law nonlinearity. The integrability aspect is the main focus of this thesis. Five different forms of nonlinearity that are considered - Kerr Law, Power Law, Parabolic Law, Dual-Power Law and Log Law nonlinearity. The traveling wave hypothesis, ansatz approach and the semi-inverse variational principle are the integrations tools that are adopted to retrieve the soliton solutions to the governing equation. As a result, several constraint conditions arise out of the integration process and represent necessary conditions for the existence of solitons.

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Chapter 1

INTRODUCTION

The dynamics of soliton propagation through optical fibers, centro-symmetric crystals, optical meta-materials and other forms of wave guides have been studied for the past few decades [1-30]. In most of the papers that are published so far, the focus is on the study of optical solitons in (1+1)-dimensions. However, it is much more practical to consider the dynamics of optical solitons in multi-dimensions. In this thesis, our study focuses on solitons in (2+1)-dimensions - two spatial and one temporal dimension.

The governing equation is the nonlinear Schrödinger equation (NLSE) with dual-dispersion term. In addition to the group-velocity dispersion (GVD), the spatio-temporal dispersion (STD) term is also taken into account. It was recently proved that the inclusion of the STD term makes the governing NLSE well-posed as opposed to NLSE with only GVD that is ill-posed [5, 17, 30].

The purpose of this paper will be to extract the exact solution to the governing NLSE. There are three integration tools that will be adopted to obtain the solution to the NLSE. The traveling wave hypothesis will reveal the waves of permanent form, while the ansatz method will also lead to the exact 1-soliton solution. In the second method, bright, dark as well as singular soliton solutions will be obtained. The third approach is the semi-inverse variational principle that will give the analytical solution to the NLSE in (2+1)-dimensions and this is the inverse problems approach. All of these integration approaches leave constraint conditions, between the parameters and coefficients, that are imperative in order for the solitons to exist. It needs to be noted that these three approaches have already been implemented to carry out the integration of the NLSE, with STD in (1+1)-dimension [3, 23, 24].

1.1 GOVERNING EQUATION

The dimensionless form of the NLSE in (2+1)-dimensions with dual-dispersion is given by [3, 23, 24]

$$iq_t + a_1\left(q_{xx} + q_{yy}\right) + a_2 q_{xt} + a_3 q_{yt} + F\left(|q|^2\right) q = 0, \tag{1.1}$$

where $q(x, y, t)$ represents the wave profile of the soliton, x and y are the spatial variables while t is the temporal variable. The first term is the linear evolution term. The coefficients of a_1 represent the GVD in the x- and y- direction. The coefficients of a_2 and a_3 represents the STDs in the x and y-directions respectively. The addition of these STD terms makes the NLSE well-posed [5, 17].

Finally, in (1.1), the last term is the nonlinear term where the functional F represents the form of nonlinearity. There are five types of nonlinearity considered. Here, F is a real-valued algebraic function.

The focus of this paper is to study the integrability of (1.1) with five forms of nonlinearity given by the functional F. These are the Kerr law, power law, parabolic law, dual-power law and the log law. There are three forms of integration architecture that are applied in this thesis. The first two approaches lead to an exact soliton solution while the third approach leads to a closed form analytical 1-soliton solution. The first method is the traveling wave hypothesis which will lead to bright 1-soliton solution only. The second form of integration, known as the ansatz approach will retrieve bright solitons, dark solitons and the singular solitons. Finally, the semi-inverse variational (SVP) approach will also only retrieve 1-soliton solution to the NLSE. These approaches are addressed in details in the next three chapters.

Chapter 2

TRAVELING WAVES

The traveling wave hypothesis leads to waves of permanent forms for equation (1.1). Occassionally, this approach is also known as the first integral method. The velocity of the soliton is obtained and 1-soliton solution will be retrieved for the five forms of nonlinearity. The starting hypothesis is that (1.1) can be written as [2, 23, 24]

$$q(x, y, t) = g(s)e^{i\phi(x,y,t)}, \tag{2.1}$$

where $g(s)$ is the amplitude component of the wave and $\phi(x, y, t)$ is the phase that is defined by

$$\phi(x, y, t) = -\kappa_1 x - \kappa_2 y + \omega t + \theta \tag{2.2}$$

and

$$s = B_1 x + B_2 y - vt. \tag{2.3}$$

Here, the parameters B_1 and B_2 are related to the inverse width of the soliton in the x- and y- directions respectively. Then the parameter v is the speed of the soliton. From the phase component, κ_j for $j = 1, 2$ gives the frequencies of the solitons along the x- and y-directions respectively while θ is the phase constant.

Substituting (2.1) into (1.1) and decomposing into real and imaginary parts leads to

$$\begin{aligned} & \left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2 B_1+a_3 B_2\right)\right\} g'' \\ - & \left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2 \kappa_1+a_3 \kappa_2-1\right)\right\} g+F\left(g^2\right) g=0 \end{aligned} \tag{2.4}$$

and

$$v=\frac{\omega\left(a_2 B_1+a_3 B_2\right)-2 a_1\left(\kappa_1 B_1+\kappa_2 B_2\right)}{1-a_2 \kappa_1-a_3 \kappa_2} \tag{2.5}$$

where g'' stands for d^2g/ds^2. Equation (2.5) poses the immediate constraint

$$a_2 \kappa_1+a_3 \kappa_2 \neq 1 \tag{2.6}$$

in order for the soliton to exist. Next, multiplying (2.4) by g' and integrating leads to

$$\begin{aligned} & \frac{B_1 x+B_2 y-v t}{\sqrt{a_1\left(B_1^2+B_2^2\right)-v\left(a_2 B_1+a_3 B_2\right)}} \\ = & \int \frac{d g}{\sqrt{\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2 \kappa_1+a_3 \kappa_2-1\right)\right\}-2 \int^g h F\left(h^2\right) d h}}, \end{aligned} \tag{2.7}$$

In the subsections that follow, equation (2.7) is used to find the 1-soliton solution for five forms of nonlinearity.

2.1 KERR LAW

This section focuses on the Kerr law of nonlinearity. The origin of the nonlinear response is related to the non-harmonic motion of bound electrons under the influence of an applied field. This results in the induced polarization being nonlinear electric field, and involves higher order terms in electric field amplitude [1].

For Kerr law nonlinearity, $F(u) = bu$, for a dummy variable u and $b \neq 0$, so then equation (1.1) reduces to

$$iq_t + a_1 \left(q_{xx} + q_{yy}\right) + a_2 q_{xt} + a_3 q_{yt} + b|q|^2 q = 0, \tag{2.8}$$

hence (2.7) simplifies to

$$\begin{aligned} &\frac{B_1 x + B_2 y - vt}{\sqrt{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2 B_1 + a_3 B_2\right)}} \\ = \;& 2\int \frac{dg}{\sqrt{2\left\{a_1\left(\kappa_1^2 + \kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2 - 1\right)\right\} - bg^4}} \end{aligned} \tag{2.9}$$

which, upon integration leads to

$$g(s) = g(B_1 x + B_2 y - vt) = A\,\mathrm{sech}[B\left(B_1 x + B_2 y - vt\right)] \tag{2.10}$$

where

$$A = \left[\frac{2\left\{a_1\left(\kappa_1^2 + \kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2 - 1\right)\right\}}{b}\right]^{\frac{1}{2}} \tag{2.11}$$

and

$$B = \left[\frac{a_1\left(\kappa_1^2 + \kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2 - 1\right)}{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2 B_1 + a_3 B_2\right)}\right]^{\frac{1}{2}}. \tag{2.12}$$

These two relations introduce the constraint conditions given by

$$b\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}>0 \tag{2.13}$$

and

$$\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}>0 \tag{2.14}$$

respectively. Finally, the 1-soliton solution to (2.8) is given by

$$q(x,y,t)=A\,\text{sech}[B\left(B_1x+B_2y-vt\right)]e^{i(-\kappa_1x-\kappa_2y+\omega t+\theta)} \tag{2.15}$$

where the amplitude A and the parameter B are respectively given by (2.11) and (2.12). An additional constraint that must also hold is given in (2.6) for the solitons to exist.

2.2 POWER LAW

Power law nonlinearity is displayed in various materials including semiconductors. This law also occurs in media for which higher order photon processes dominate at different intensities. Moreover, in nonlinear plasmas, the power law solves the problem of small K-condensation in weak turbulence theory and this law is also viewed as a generalization to the Kerr law nonlinearity [1]. For power law nonlinearity $F(u)=bu^n$, where $b\neq 0$, the NLSE given by

$$iq_t+a_1\left(q_{xx}+q_{yy}\right)+a_2q_{xt}+a_3q_{yt}+b|q|^{2n}q=0, \tag{2.16}$$

reduces to Kerr law NLSE given by (2.7). It must be noted that for practical considerations,

$$0 < n < 2, \tag{2.17}$$

in particular $n \neq 2$ in order to avoid self-focusing singularity [1, 10]. Therefore equation (2.6), becomes

$$\begin{aligned} & \frac{B_1 x + B_2 y - vt}{\sqrt{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2 B_1 + a_3 B_2\right)}} \\ = \; & (n+1) \int \frac{dg}{\sqrt{(n+1)\left\{a_1\left(\kappa_1^2 + \kappa_2^2\right) - \omega\left(a_2 \kappa_1 + a_3 \kappa_2 - 1\right)\right\} - b g^{2n+2}}} \end{aligned} \tag{2.18}$$

and yields the solution

$$g(s) = g(B_1 x + B_2 y - vt) = A \operatorname{sech}^{\frac{1}{n}}\left[B\left(B_1 x + B_2 y - vt\right)\right] \tag{2.19}$$

with

$$A = \left[\frac{(n+1)\left\{a_1\left(\kappa_1^2 + \kappa_2^2\right) - \omega\left(a_2 \kappa_1 + a_3 \kappa_2 - 1\right)\right\}}{b}\right]^{\frac{1}{2n}} \tag{2.20}$$

and

$$B = n\left[\frac{a_1\left(\kappa_1^2 + \kappa_2^2\right) - \omega\left(a_2 \kappa_1 + a_3 \kappa_2 - 1\right)}{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2 B_1 + a_3 B_2\right)}\right]^{\frac{1}{2}} \tag{2.21}$$

Requiring that $A > 0$ and $B > 0$, lead to (2.13) and (2.14), the same constraints as given for Kerr law nonlinearity. The 1-soliton solution is

$$q(x,y,t) = A\ \mathrm{sech}^{\frac{1}{n}}[B\left(B_1x + B_2y - vt\right)]e^{i(-\kappa_1 x - \kappa_2 y + \omega t + \theta)} \tag{2.22}$$

which reduces to the soliton for the Kerr law nonlinearity when $n = 1$.

2.3 PARABOLIC LAW

It is necessary to consider nonlinearities higher than the third-order in order in an attempt to obtain some knowledge of the diameter of a self-trapping beam. There was little or no attention paid to the propagation of optical beams in the fifth-order nonlinear media, since no analytic solutions existed and it seemed that chances of finding any material with significant fifth-order term was low, until the present time [1, 9]. For parabolic law nonlinearity, also known as cubic-quintic law, $F(u) = b_1u + b_2u^2$, with $b_1 \neq 0$ and $b_2 \neq 0$, the NLSE becomes

$$iq_t + a_1\left(q_{xx} + q_{yy}\right) + a_2q_{xt} + a_3q_{yt} + \left(b_1\left|q\right|^2 + b_2\left|q\right|^4\right)q = 0, \tag{2.23}$$

and (2.7) reduces to

$$\begin{aligned} &\frac{B_1x + B_2y - vt}{\sqrt{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2B_1 + a_3B_2\right)}} \\ = &\int \frac{\sqrt{6}dg}{\sqrt{6\left\{a_1\left(\kappa_1^2 + \kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2 - 1\right)\right\} - 3b_1g^4 - 2b_2g^6}}, \end{aligned} \tag{2.24}$$

and leads to

$$g(s) = g(B_1x + B_2y - vt) = \frac{A}{\{D_1 + \cosh\left[B\left(B_1x + B_2y - vt\right)\right]\}^{\frac{1}{2}}}. \tag{2.25}$$

The amplitude is

$$A = \frac{2\sqrt{3\left\{a_1\left(\kappa_1^2 + \kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2 - 1\right)\right\}}}{\left[9b_1^2 + 48b_2\left\{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2B_1 + a_3B_2\right)\right\}\right]^{\frac{1}{4}}}, \tag{2.26}$$

while the parameters B and D_1 are respectively given by

$$B = 2\left[\frac{a_1\left(\kappa_1^2 + \kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2 - 1\right)}{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2B_1 + a_3B_2\right)}\right]^{\frac{1}{2}} \tag{2.27}$$

and

$$D_1 = \frac{\sqrt{3}b_1}{\sqrt{3b_1^2 + 16b_2\left\{a_1\left(\kappa_1^2 + \kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2 - 1\right)\right\}}}. \tag{2.28}$$

For the solitons to exist, however, the form of the expressions for the parameters A, B and D_1 we deduce that the following constraints must be satisfied

$$a_1\left(B_1^2 + B_2^2\right) - v\left(a_2B_1 + a_3B_2\right) > 0, \tag{2.29}$$

$$a_1\left(\kappa_1^2 + \kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2 - 1\right) > 0, \tag{2.30}$$

$$3b_1^2 + 16b_2\left\{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2B_1 + a_3B_2\right)\right\} > 0. \tag{2.31}$$

The 1-soliton solution to the NLSE with parabolic law nonlinearity is

$$q(x,y,t) = \frac{A}{\{D_1 + \cosh\left[B\left(B_1x + B_2y - vt\right)\right]\}^{\frac{1}{2}}} e^{i(-\kappa_1 x - \kappa_2 y + \omega t + \theta)} \tag{2.32}$$

where all the parameters are now defined along with the constraint conditions (2.6), (2.29)-(2.31).

2.4 DUAL-POWER LAW

This model is a generalized version of the parabolic law nonlinearity and it also serves as a basic model to describe solitons in photovoltaic-photorefractive materials such as $LiNbO_3$ [10]. Here, $F(u) = b_1 u^n + b_2 u^{2n}$, with b_1 and b_2 being non-zero constants, and consequently, if $n = 1$ the dual-power law nonlinearity collapses to parabolic law. The NLSE is now given by

$$iq_t + a_1\left(q_{xx} + q_{yy}\right) + a_2 q_{xt} + a_3 q_{yt} + \left(b_1 |q|^{2n} + b_2 |q|^{4n}\right) q = 0, \tag{2.33}$$

and therefore (2.7) reduces to

$$\frac{B_1x + B_2y - vt}{\sqrt{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2B_1 + a_3B_2\right)}} = \int \frac{\sqrt{(n+1)(2n+1)}dg}{\sqrt{(n+1)(2n+1)\left\{a_1\left(\kappa_1^2 + \kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2 - 1\right)\right\} - (2n+1)b_1 g^{2n+2} - (n+1)b_2 g^{4n+2}}} \tag{2.34}$$

and leads to

$$g(s) = g(B_1x + B_2y - vt) = \frac{A}{\{D_2 + \cosh\left[B\left(B_1x + B_2y - vt\right)\right]\}^{\frac{1}{2n}}} \tag{2.35}$$

where the amplitude is

$$A = \frac{(n+1)\sqrt{(2n+1)\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}}}{\left[(2n+1)^2b_1^2+4(n+1)^2(2n+1)b_2\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}\right]^{\frac{1}{4}}}, \tag{2.36}$$

and the parameters B and D_2 are

$$B = 2n\left[\frac{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)}{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)}\right]^{\frac{1}{2}} \tag{2.37}$$

and

$$D_2 = \frac{b_1\sqrt{2n+1}}{\sqrt{(2n+1)b_1^2+4b_2(n+1)^2\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}}}. \tag{2.38}$$

In this case, D_2 poses the constraint

$$(2n+1)b_1^2+4(n+1)^2b_2\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}>0. \tag{2.39}$$

Therefore, the 1-soliton solution to the NLSE with dual-power law nonlinearity is

$$q(x,y,t) = \frac{A}{\left\{D_2+\cosh\left[B\left(B_1x+B_2y-vt\right)\right]\right\}^{\frac{1}{2n}}}e^{i(-\kappa_1x-\kappa_2y+\omega t+\theta)}, \tag{2.40}$$

where the parameters are all defined and the constraint conditions are given by (2.6), (2.29), (2.30) and (2.39). One can clearly observe that all the results of this subsection collapses to the results of the parabolic law, when $n=1$.

2.5 LOG LAW

This law appears in contemporary physics. It allows for closed form exact expressions for Gaussian beams (*Gaussons*) as well as in the periodic and quasi-periodic regimes of beam evolution. The advantage of this model is that the beams do not shed any radiation and this gives log law nonlinearity an edge over other forms of nonlinearity. For log law nonlinearity, $F(u) = b \ln u$, with $b \neq 0$, the governing NLSE is [3]

$$iq_t + a_1 \left(q_{xx} + q_{yy}\right) + a_2 q_{xt} + a_3 q_{yt} + bq \ln |q|^2 = 0. \tag{2.41}$$

Therefore (2.7), in this case, simplifies to

$$\frac{B_1 x + B_2 y - vt}{\sqrt{a_1 \left(B_1^2 + B_2^2\right) - v \left(a_2 B_1 + a_3 B_2\right)}} = \int \frac{dg}{g\sqrt{\left\{a_1 \left(\kappa_1^2 + \kappa_2^2\right) - \omega \left(a_2 \kappa_1 + a_3 \kappa_2 - 1\right)\right\} - b\left(2 \ln g - 1\right)}}, \tag{2.42}$$

which integrates to

$$g(s) = g(B_1 x + B_2 y - vt) = Ae^{-B^2 (B_1 x + B_2 y - vt)^2}, \tag{2.43}$$

where the amplitude is

$$A = \exp\left\{\frac{a_1 \left(\kappa_1^2 + \kappa_2^2\right) - \omega \left(a_2 \kappa_1 + a_3 \kappa_2 - 1\right) + b}{2b}\right\} \tag{2.44}$$

and the parameter B is

$$B = \sqrt{\frac{b}{2\left\{a_1 \left(B_1^2 + B_2^2\right) - v \left(a_2 B_1 + a_3 B_2\right)\right\}}}. \tag{2.45}$$

The existence of the Gaussons are guaranteed through the constraint conditions (7) and

$$b\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}>0 \tag{2.46}$$

Finally, the 1-soliton, or Gausson, solution to (42) is given by

$$q(x,y,t)=Ae^{-B^2(B_1x+B_2y-vt)^2}e^{i(-\kappa_1x-\kappa_2y+\omega t+\theta)} \tag{2.47}$$

where the parameters are all defined with the necessary constraint conditions in place.

Chapter 3

ANSATZ METHOD

This chapter discusses the second integration tool that we adopt to integrate the NLSE with STD. This method typically reveals three types of solitons, bright solitons, dark solitons and singular optical solitons. There are the necessary constraint conditions that will naturally arise from the structure of the soliton solution. Thus, this method has an edge over the traveling wave hypothesis approach.

The starting hypothesis for the ansatz method for NLSE given by (1.1) is [23, 24]

$$q(x,t) = P(x,y,t)e^{i\phi(x,y,t)} \tag{3.1}$$

where $P(x,y,t)$ represents the amplitude component of the soliton and it is assumed that P is at least twice differentiable with respect to its variables. Therefore, substituting (3.1) into (1.1) and decomposing into real and imaginary parts leads to

$$\begin{aligned} &\left\{\omega\left(1 - a_2\kappa_1 - a_3\kappa_2\right) + a_1\left(\kappa_1^2 + \kappa_2^2\right)\right\} P - F\left(P^2\right) P \\ - \quad & a_1\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right) - a_2\frac{\partial^2 P}{\partial x\partial t} - a_3\frac{\partial^2 P}{\partial y\partial t} = 0 \end{aligned} \tag{3.2}$$

and

$$\left(1 - a_2\kappa_1 - a_3\kappa_2\right)\frac{\partial P}{\partial t} - 2a_1\left(\kappa_1\frac{\partial P}{\partial x} + \kappa_2\frac{\partial P}{\partial y}\right) + \omega\left(a_2\frac{\partial P}{\partial x} + a_3\frac{\partial P}{\partial y}\right) = 0 \tag{3.3}$$

The imaginary part equation, will lead to the speed of the soliton as determined earlier in (2.5). This rest of this chapter will focus on the real part of the equation given by (3.2). The study will now be split into three subsections. These subsections will address bright solitons, dark solitons and singular solitons respectively.

3.1 BRIGHT SOLITONS

The bright solitons are derived for five types of nonlinearity as in the previous section.

3.1.1 KERR LAW

For Kerr law nonlinearity, the real part of equation (3.2) reduces to

$$\begin{aligned} &\left\{\omega\left(1-a_2\kappa_1-a_3\kappa_2\right)+a_1\left(\kappa_1^2+\kappa_2^2\right)\right\}P-bP^3 \\ -&\ a_1\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}\right)-a_2\frac{\partial^2 P}{\partial x\partial t}-a_3\frac{\partial^2 P}{\partial y\partial t}=0. \end{aligned} \tag{3.4}$$

The starting hypothesis in this case will be [23, 24]

$$P(x,y,t)=A\,\mathrm{sech}^p s \tag{3.5}$$

where s is the same as defined in (2.3) and the exponent p is unknown at this point. The value of this unknown exponent p is revealed, once the balancing principle is implemented. Substituting (3.5) into (3.4) gives

$$\begin{aligned} &\left\{\omega+a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2\right)\right\}\mathrm{sech}^p s-bA^2\,\mathrm{sech}^{3p}s \\ +&\ \left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}\left\{p^2\,\mathrm{sech}^p s-p(p+1)\,\mathrm{sech}^{p+2}s\right\}=0. \end{aligned} \tag{3.6}$$

By the balancing principle, equating the exponents $3p$ and $p+2$ leads to

$$p = 1. \tag{3.7}$$

Next, setting the coefficients of the linearly independent functions $\text{sech}^{p+j}s$ to zero, for $j = 0, 2$ leads to

$$\omega = \frac{v(a_2B_1 + a_3B_2) - a_1(B_1^2 + B_2^2 + \kappa_1^2 + \kappa_2^2)}{1 - a_2\kappa_1 - a_3\kappa_2} \tag{3.8}$$

and

$$A = \left[\frac{2\{v(a_2B_1 + a_3B_2) - a_1(B_1^2 + B_2^2)\}}{b}\right]^{\frac{1}{2}}, \tag{3.9}$$

~~provided~~

$$b\left\{v(a_2B_1 + a_3B_2) - a_1\left(B_1^2 + B_2^2\right)\right\} > 0 \tag{3.10}$$

and (2.6) remain valid. Thus, the 1-soliton solution to the NLSE by ansatz method is given by

$$q(x,y,t) = A\,\text{sech}\,(B_1x + B_2y - vt)\, e^{i(-\kappa_1x - \kappa_2y + \omega t + \theta)}. \tag{3.11}$$

3.1.2 POWER LAW

For power law nonlinearity, equation (3.2) transforms to

$$\begin{aligned} &\left\{\omega(1 - a_2\kappa_1 - a_3\kappa_2) + a_1\left(\kappa_1^2 + \kappa_2^2\right)\right\}P - bP^{2n+1} \\ -\ &a_1\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right) - a_2\frac{\partial^2 P}{\partial x \partial t} - a_3\frac{\partial^2 P}{\partial y \partial t} = 0. \end{aligned} \tag{3.12}$$

so that the starting hypothesis stays the same as in (3.5). Therefore substituting (3.5) into (3.12) leads to

$$\begin{aligned} & \left\{\omega + a_1\left(\kappa_1^2+\kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2\right)\right\} \operatorname{sech}^p s - bA^{2n}\operatorname{sech}^{(2n+1)p} s \\ + & \left\{a_1\left(B_1^2+B_2^2\right) - v\left(a_2B_1+a_3B_2\right)\right\}\left\{p^2\operatorname{sech}^p s - p(p+1)\operatorname{sech}^{p+2} s\right\} = 0. \end{aligned} \tag{3.13}$$

Again, applying the balancing principle yields

$$p = \frac{1}{n}. \tag{3.14}$$

Equating the coefficients of the same linearly independent functions, as for the Kerr law case, implies

$$\omega = \frac{v\left(a_2B_1+a_3B_2\right) - a_1\left\{\left(B_1^2+B_2^2\right) + n^2\left(\kappa_1^2+\kappa_2^2\right)\right\}}{1-a_2\kappa_1-a_3\kappa_2} \tag{3.15}$$

and

$$A = \left[\frac{(n+1)\left\{v\left(a_2B_1+a_3B_2\right) - a_1\left(B_1^2+B_2^2\right)\right\}}{n^2 b}\right]^{\frac{1}{2n}} \tag{3.16}$$

which leads to the same constraint condition (3.10). Therefore, the 1-soliton solution with the power law nonlinearity is given by

$$q(x,y,t) = A\operatorname{sech}^{\frac{1}{n}}\left(B_1x+B_2y-vt\right)e^{i(-\kappa_1x-\kappa_2y+\omega t+\theta)}. \tag{3.17}$$

3.1.3 PARABOLIC LAW

For this case, equation (3.2) transforms to

$$
\begin{aligned}
& \left\{\omega\left(1-a_2\kappa_1-a_3\kappa_2\right)+a_1\left(\kappa_1^2+\kappa_2^2\right)\right\}P-b_1P^3-b_2P^5 \\
- & \; a_1\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}\right)-a_2\frac{\partial^2 P}{\partial x\partial t}-a_3\frac{\partial^2 P}{\partial y\partial t}=0.
\end{aligned}
\tag{3.18}
$$

The starting hypothesis is given by [23, 24]

$$
P(x,y,t)=\frac{A}{\left(D_1+\cosh s\right)^p}
\tag{3.19}
$$

where, D_1 is a parameters whose value and that of the exponent p is determined by using the balancing principle. Substituting (3.19) into (3.18) leads to

$$
\begin{aligned}
& \left[\omega+a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2\right)\right. \\
- & \left. p^2\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}\right] \\
+ & \; p(2p+1)D_1\frac{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)}{D_1+\cosh s} \\
+ & \; p(p+1)\left(D_1^2-1\right)\frac{v\left(a_2B_1+a_3B_2\right)-a_1\left(B_1^2+B_2^2\right)}{\left(D_1+\cosh s\right)^2} \\
- & \; \frac{b_1A^2}{\left(D_1+\cosh s\right)^{2p}}-\frac{b_2A^4}{\left(D_1+\cosh s\right)^{4p}}=0
\end{aligned}
\tag{3.20}
$$

By the balancing the pairs of exponent ($2p$, 1) or ($4p$, 2) leads to

$$
p=\frac{1}{2}.
\tag{3.21}
$$

Finally, setting the coefficients of the linearly independent functions $1/(D_1 + \cosh s)^j$ to zero, for $j = 0, 1, 2$ leads to

$$\omega = \frac{v\left(a_2 B_1 + a_3 B_2\right) - a_1\left(B_1^2 + B_2^2\right) - 4a_1\left(\kappa_1^2 + \kappa_2^2\right)}{4\left\{1 - \left(a_2\kappa_1 + a_3\kappa_2\right)\right\}} \tag{3.22}$$

and the amplitude is given by

$$A = \left[\frac{D_1\left\{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2 B_1 + a_3 B_2\right)\right\}}{b_1}\right]^{\frac{1}{2}}, \tag{3.23}$$

where the parameter

$$D_1 = b_1\sqrt{\frac{3}{3b_1^2 + 4b_2\left\{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2 B_1 + a_3 B_2\right)\right\}}}. \tag{3.24}$$

The constraint relation is

$$3b_1^2 + 4b_2\left\{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2 B_1 + a_3 B_2\right)\right\} > 0 \tag{3.25}$$

and as before (2.6) must still remain valid. Hence, the 1-soliton solution to the NLSE with parabolic law nonlinearity is

$$q(x, y, t) = \frac{A}{\left\{D_1 + \cosh\left(B_1 x + B_2 y - vt\right)\right\}^{\frac{1}{2}}} e^{i(-\kappa_1 x - \kappa_2 y + \omega t + \theta)} \tag{3.26}$$

3.1.4 DUAL-POWER LAW

From (3.2), the function $P(x, t)$ satisfies

$$\begin{aligned} &\left\{\omega\left(1 - a_2\kappa_1 - a_3\kappa_2\right) + a_1\left(\kappa_1^2 + \kappa_2^2\right)\right\} P - b_1 P^{2n+1} - b_2 P^{4n+1} \\ - \; &a_1\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right) - a_2\frac{\partial^2 P}{\partial x \partial t} - a_3\frac{\partial^2 P}{\partial y \partial t} = 0. \end{aligned} \tag{3.27}$$

The trial function is given by [23, 24]

$$P(x,y,t)=\frac{A}{(D_2+\cosh s)^p} \tag{3.28}$$

where D_2 is a parameter. Substituting (3.28) into (3.27) leads to

$$\begin{aligned}
&\left[\omega+a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2\right)\right.\\
-&\left.p^2\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}\right]\\
+&\ p(2p+1)D_2\frac{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)}{D_2+\cosh s}\\
+&\ p(p+1)\left(D_2^2-1\right)\frac{v\left(a_2B_1+a_3B_2\right)-a_1\left(B_1^2+B_2^2\right)}{(D_2+\cosh s)^2}\\
-&\ \frac{b_1A^{2n}}{(D_2+\cosh s)^{2np}}-\frac{b_2A^{4n}}{(D_2+\cosh s)^{4np}}=0.
\end{aligned} \tag{3.29}$$

Again, by the balancing principle, from the exponents pairs ($2np$, 1) or ($4np$, 2) it is found that

$$p=\frac{1}{2n}. \tag{3.30}$$

Equating the coefficients of the linearly independent functions of $1/(D_1+\cosh s)^j$ to zero, for $j=0,1,2$ the following expressions are obtained for soliton parameters ω, A and D_2:

$$\omega=\frac{v\left(a_2B_1+a_3B_2\right)-a_1\left(B_1^2+B_2^2\right)-4n^2a_1\left(\kappa_1^2+\kappa_2^2\right)}{4n^2\left\{1-\left(a_2\kappa_1+a_3\kappa_2\right)\right\}}, \tag{3.31}$$

$$A=\left[\frac{(n+1)D_2\left\{v\left(a_2B_1+a_3B_2\right)-a_1\left(B_1^2+B_2^2\right)\right\}}{2n^2b_1}\right]^{\frac{1}{2n}}, \tag{3.32}$$

$$D_2 = nb_1\sqrt{\frac{2n+1}{n^2(2n+1)b_1^2 + (n+1)^2 b_2 \left\{a_1\left(B_1^2+B_2^2\right) - v\left(a_2B_1 + a_3B_2\right)\right\}}}. \tag{3.33}$$

The constraint relation, in this case, is

$$n^2(2n+1)b_1^2 + (n+1)^2 b_2 \left\{a_1\left(B_1^2+B_2^2\right) - v\left(a_2B_1 + a_3B_2\right)\right\} > 0. \tag{3.34}$$

As before, (2.6) must remain valid as well. It needs to be noted that all the results in this subsection for dual-power law nonlinearity collapses to the results of the previous section for parabolic law nonlinearity upon setting $n = 1$. Hence, the 1-soliton solution to the NLSE with dual-power law is given by

$$q(x,y,t) = \frac{A}{\left\{D_2 + \cosh\left(B_1x + B_2y - vt\right)\right\}^{\frac{1}{2n}}} e^{i(-\kappa_1 x - \kappa_2 y + \omega t + \theta)}. \tag{3.35}$$

3.1.5 LOG LAW

For log law nonlinearity, the real part equation given by (3.2) leads to

$$\begin{aligned} &\left\{\omega\left(1 - a_2\kappa_2 - a_3\kappa_2\right) + a_1\left(\kappa_1^2 + \kappa_2^2\right)\right\} P - bP\ln P^2 \\ - &\ a_1\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right) - a_2\frac{\partial^2 P}{\partial x \partial t} - a_3\frac{\partial^2 P}{\partial y \partial t} = 0. \end{aligned} \tag{3.36}$$

so that the starting hypothesis is given by

$$P(x,y,t) = Ae^{-s^2} \tag{3.37}$$

which is a Gausson beam with the amplitude A and widths B_1 and B_2 along x- and y- directions respectively. Substituting (3.37) into (3.36) leads to

$$\begin{aligned} & \omega\left\{1-\left(a_2\kappa_1+a_3\kappa_2\right)\right\}+a_1\left\{\left(\kappa_1^2+\kappa_2^2\right)+2\left(B_1^2+B_2^2\right)\right\} \\ - & 2\left\{v\left(a_2B_1+a_3B_2\right)+b\ln A\right\} \\ - & 2\left\{2a_1\left(B_1^2+B_2^2\right)-2v\left(a_2B_1+a_3B_2\right)-b\right\}s^2=0. \end{aligned} \tag{3.38}$$

Next, setting the coefficients of the linearly independent functions s^j for $j=0,2$ to zero leads to

$$\omega=\frac{2b\ln A+2v\left(a_2B_1+a_3B_2\right)-2a_1\left(B_1^2+B_2^2\right)-a_1\left(\kappa_1^2+\kappa_2^2\right)}{1-\left(a_2\kappa_1+a_3\kappa_2\right)} \tag{3.39}$$

and

$$2a_1\left(B_1^2+B_2^2\right)=2v\left(a_2B_1+a_3B_2\right)+b \tag{3.40}$$

which connects the two widths of the Gausson with its speed. Thus, finally, the Gausson beam solution to the NLSE with log law nonlinearity is given by

$$q(x,y,t)=Ae^{-(B_1x+B_2y-vt)^2}e^{i(-\kappa_1x-\kappa_2y+\omega t+\theta)} \tag{3.41}$$

with the constraint (2.6) in place.

3.2 DARK SOLITONS

In this subsection we obtain the dark soliton solution to the NLSE with STD in (2+1)-dimensions. It is not yet known if log law nonlinearity supports dark solitons.

Therefore the study will be only with Kerr law, power law, parabolic and dual-power laws of nonlinearity.

3.2.1 KERR LAW

For Kerr law nonlinearity, the starting hypothesis is given by [23, 24]

$$P(x,y,t) = A\tanh^p s \tag{3.42}$$

where the value of the unknown exponent p is determined by the balancing principle and the parameters s is still defined by (2.3). It needs to be noted that for dark solitons the parameters A, B_1 and B_2 are all free parameters. Now, substituting (3.42) into (3.4) leads to

$$\begin{aligned}
&\left\{\omega\left(1-a_2\kappa_1-a_3\kappa_2\right)+a_1\left(\kappa_1^2+\kappa_2^2\right)\right\}\tanh^p s - bA^2\tanh^{3p} s \\
-\ & p\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}\times \\
\times\ & \left\{(p-1)\tanh^{p-2} s - 2p\tanh^p s + (p+1)\tanh^{p+2} s\right\} = 0
\end{aligned} \tag{3.43}$$

By the balancing principle as for the bright solitons, $p = 1$. Next, setting the coefficients of the remaining linearly independent functions $\tanh^{p+j} s$ for $j = 0, 2$ leads to

$$\omega = \frac{2\left\{v\left(a_2B_1+a_3B_2\right)-a_1\left(B_1^2+B_2^2\right)\right\}-a_1\left(\kappa_1^2+\kappa_2^2\right)}{1-a_2\kappa_1-a_3\kappa_2} \tag{3.44}$$

and

$$A = \left[\frac{2\left\{v\left(a_2B_1+a_3B_2\right)-a_1\left(B_1^2+B_2^2\right)\right\}}{b}\right]^{\frac{1}{2}} \tag{3.45}$$

provided

$$b\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}<0. \tag{3.46}$$

The dark 1-soliton solution to the NLSE

$$q(x,y,t)=A\tanh\left(B_1x+B_2y-vt\right)e^{i(-\kappa_1x-\kappa_2y+\omega t+\theta)} \tag{3.47}$$

with the constraint conditions given by (2.6) and (3.46).

3.2.2 POWER LAW

For power law nonlinearity, substituting (3.42) into (3.12) leads to

$$\begin{aligned}
&\left\{\omega\left(1-a_2\kappa_1-a_3\kappa_2\right)+a_1\left(\kappa_1^2+\kappa_2^2\right)\right\}\tanh^p s-bA^{2n}\tanh^{(2n+1)p}s\\
-\;&p\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}\times\\
\times\;&\left\{(p-1)\tanh^{p-2}s-2p\tanh^p s+(p+1)\tanh^{p+2}s\right\}=0.
\end{aligned} \tag{3.48}$$

By the balancing principle, the value of $p=1/n$ and thus

$$n=1. \tag{3.49}$$

This means that the NLSE with power law nonlinearity collapses to NLSE with Kerr law nonlinearity. Hence, the NLSE with power law nonlinearity does not support dark soliton solution unless it reduces to Kerr law nonlinearity. This is a very important observation that is being made for the first time, in the context of NLSE in (2+1)-dimensions with STD. Therefore, all the results from the previous subsection hold true for power law NLSE. Hence equations (3.44)-(3.47) are also valid for power law nonlinearity.

3.2.3 PARABOLIC LAW

For parabolic law nonlinearity, we assume dark soliton is given by [23, 24, 27]

$$q(x,y,t) = (A + B\tanh s)^p e^{i\phi}. \tag{3.50}$$

Substituting (3.50) into (2.23) and decomposing into real and imaginary parts leads to

$$\begin{aligned}
- & \omega (A + B\tanh s)^p + a_1 B^2 \left(B_1^2 + B_2^2\right) p\,(p-1)\left(1 - \frac{A^2}{B^2}\right)^2 (A + B\tanh s)^{p-2} \\
+ & \frac{a_1 p\,(p+1)\left(B_1^2 + B_2^2\right)}{B^2} (A + B\tanh s)^{p+2} \\
+ & 2a_1 p\,(2p-1)\,A\left(B_1^2 + B_2^2\right)\left(1 - \frac{A^2}{B^2}\right)(A + B\tanh s)^{p-1} \\
+ & a_1\left[2\left(B_1^2 + B_2^2\right)p^2\left(\frac{3A^2}{B^2} - 1\right) - \kappa_1^2 - \kappa_2^2\right](A + B\tanh s)^p \\
- & \frac{2a_1 p\,(2p+1)\,A\left(B_1^2 + B_2^2\right)}{B^2} (A + B\tanh s)^{p+1} \\
- & B^2\left(a_2 B_1 + a_3 B_2\right) v p\,(p-1)\left(1 - \frac{A^2}{B^2}\right)^2 (A + B\tanh s)^{p-2} \\
- & \frac{p\,(p+1)\left(a_2 B_1 + a_3 B_2\right) v}{B^2} (A + B\tanh s)^{p+2} \\
- & 2p\,(2p-1)\left(1 - \frac{A^2}{B^2}\right) A\left(a_2 B_1 + a_3 B_2\right) v\,(A + B\tanh s)^{p-1} \\
+ & \left[2\left(a_2 B_1 + a_3 B_2\right) v p^2\left(1 - \frac{3A^2}{B^2}\right) + \left(a_2\kappa_1 + a_3\kappa_2\right)\omega\right](A + B\tanh s)^p \\
+ & \frac{2p\,(2p+1)\,A\left(a_2 B_1 + a_3 B_2\right) v}{B^2} (A + B\tanh s)^{p+1} \\
+ & b_1 (A + B\tanh s)^{3p} + b_2 (A + B\tanh s)^{5p} = 0
\end{aligned} \tag{3.51}$$

and

$$\begin{aligned}
&- \ pBv\left(1-\frac{A^2}{B^2}\right)(A+B\tanh s)^{p-1} - \frac{2pvA}{B}(A+B\tanh s)^{p} \\
&+ \ \frac{pv}{B}(A+B\tanh s)^{p+1} \\
&- \ 2a_1pB\left(\kappa_1B_1+\kappa_2B_2\right)\left(1-\frac{A^2}{B^2}\right)(A+B\tanh s)^{p-1} \\
&- \ \frac{4a_1pA\left(\kappa_1B_1+\kappa_2B_2\right)}{B}(A+B\tanh s)^{p} \\
&+ \ \frac{2a_1p\left(\kappa_1B_1+\kappa_2B_2\right)}{B}(A+B\tanh s)^{p+1} \\
&+ \ pB\left(1-\frac{A^2}{B^2}\right)\left[(a_2\kappa_1+a_3\kappa_2)v+\omega(a_2B_1+a_3B_2)\right](A+B\tanh s)^{p-1} \\
&+ \ \frac{2pA}{B}\left[(a_2\kappa_1+a_3\kappa_2)v+\omega(a_2B_1+a_3B_2)\right](A+B\tanh s)^{p} \\
&- \ \frac{p}{B}\left[(a_2\kappa_1+a_3\kappa_2)v+\omega(a_2B_1+a_3B_2)\right](A+B\tanh s)^{p+1} = 0
\end{aligned} \tag{3.52}$$

Equating the exponent pairs $(p+2, 5p)$ or $(p+1, 3p)$ lead to $p = 1/2$. From equation (3.52), setting the coefficients of the linearly independent functions $(A + B\tanh s)^{p+j}$ to zero, for $j = 0, \pm 1$, gives

$$A = B, \tag{3.53}$$

and the speed of the soliton that is given by (2.5). Now from equation (3.51), the following relations are recovered after setting the coefficients of the linearly independent functions $(A + B\tanh s)^{p+j}$ to zero, where $j = 0, \pm 1, \pm 2$, gives

$$\omega = \frac{a_1\left(\kappa_1^2+\kappa_2^2\right)}{a_2\kappa_1+a_3\kappa_2-1}, \tag{3.54}$$

$$v = \frac{a_1\left(B_1^2+B_2^2\right)}{\left(a_2B_1+a_3B_2\right)}, \tag{3.55}$$

and

$$A = B = -\frac{3b_1}{8b_2}. \tag{3.56}$$

These relations prompt the constraints

$$a_2B_1 + a_3B_2 \neq 0, \tag{3.57}$$

and

$$b_2 \neq 0. \tag{3.58}$$

The dark 1-soliton solution for the NLSE with parabolic law nonlinearity is given by

$$q(x, y, t) = \sqrt{A\left\{1 + \tanh\left(B_1x + B_2y - vt\right)\right\}}e^{i(-\kappa_1x - \kappa_2y + \omega t + \theta)}. \tag{3.59}$$

3.2.4 DUAL-POWER LAW

For dual-power law nonlinearity, our starting hypothesis for dark 1-soliton solution, stays the same as given by (3.50) [23, 24, 27]. Upon substituting this ansatz into

(2.33) and decomposing into real and imaginary parts leads to

$$
\begin{aligned}
- & \ \omega\,(A+B\tanh s)^p + a_1 B^2\left(B_1^2+B_2^2\right)p\,(p-1)\left(1-\frac{A^2}{B^2}\right)^2 (A+B\tanh s)^{p-2} \\
+ & \ \frac{a_1 p\,(p+1)\left(B_1^2+B_2^2\right)}{B^2}(A+B\tanh s)^{p+2} \\
+ & \ 2a_1 p\,(2p-1)\,A\left(B_1^2+B_2^2\right)\left(1-\frac{A^2}{B^2}\right)(A+B\tanh s)^{p-1} \\
+ & \ a_1\left[2\left(B_1^2+B_2^2\right)p^2\left(\frac{3A^2}{B^2}-1\right)-\kappa_1^2-\kappa_2^2\right](A+B\tanh s)^p \\
- & \ \frac{2a_1 p\,(2p+1)\,A\left(B_1^2+B_2^2\right)}{B^2}(A+B\tanh s)^{p+1} \\
- & \ B^2\left(a_2B_1+a_3B_2\right)vp\,(p-1)\left(1-\frac{A^2}{B^2}\right)^2(A+B\tanh s)^{p-2} \\
- & \ \frac{p\,(p+1)\left(a_2B_1+a_3B_2\right)v}{B^2}(A+B\tanh s)^{p+2} \\
- & \ 2p\,(2p-1)\left(1-\frac{A^2}{B^2}\right)A\left(a_2B_1+a_3B_2\right)v\,(A+B\tanh\tau)^{p-1} \\
+ & \ \left[2\left(a_2B_1+a_3B_2\right)vp^2\left(1-\frac{3A^2}{B^2}\right)+\left(a_2\kappa_1+a_3\kappa_2\right)\omega\right](A+B\tanh s)^p \\
+ & \ \frac{2p\,(2p+1)\,A\left(a_2B_1+a_3B_2\right)v}{B^2}(A+B\tanh s)^{p+1} \\
+ & \ b_1\,(A+B\tanh s)^{p(2n+1)}+b_2\,(A+B\tanh s)^{p(4n+1)}=0,
\end{aligned}
\tag{3.60}
$$

and

$$
\begin{aligned}
&- \quad pBv\left(1-\frac{A^2}{B^2}\right)(A+B\tanh s)^{p-1} \\
&- \quad \frac{2pvA}{B}(A+B\tanh s)^{p}+\frac{pv}{B}(A+B\tanh s)^{p+1} \\
&- \quad 2a_1pB\left(\kappa_1B_1+\kappa_2B_2\right)\left(1-\frac{A^2}{B^2}\right)(A+B\tanh s)^{p-1} \\
&- \quad \frac{4a_1pA\left(\kappa_1B_1+\kappa_2B_2\right)}{B}(A+B\tanh s)^{p} \\
&+ \quad \frac{2a_1p\left(\kappa_1B_1+\kappa_2B_2\right)}{B}(A+B\tanh s)^{p+1} \\
&+ \quad pB\left(1-\frac{A^2}{B^2}\right)\left[\left(a_2\kappa_1+a_3\kappa_2\right)v+\omega\left(a_2B_1+a_3B_2\right)\right](A+B\tanh s)^{p-1} \\
&+ \quad \frac{2pA}{B}\left[\left(a_2\kappa_1+a_3\kappa_2\right)v+\omega\left(a_2B_1+a_3B_2\right)\right](A+B\tanh s)^{p} \\
&- \quad \frac{p}{B}\left[\left(a_2\kappa_1+a_3\kappa_2\right)v+\omega\left(a_2B_1+a_3B_2\right)\right](A+B\tanh s)^{p+1}=0 \qquad (3.61)
\end{aligned}
$$

respectively. From (3.60), equating the exponents $(p+1,\, p(2n+1))$ leads to the same value of p as in (3.30). From the linearly independent functions in (3.61), the speed of the soliton is retrieved as in (2.5) and the same relation (3.53) is obtained. Next, from the real part equation given by (3.60), the following relations are obtained

$$\omega=\frac{a_1\left(\kappa_1^2+\kappa_2^2\right)}{a_2\kappa_1+a_3\kappa_2-1}, \qquad (3.62)$$

$$v=\frac{a_1\left(B_1^2+B_2^2\right)}{\left(a_2B_1+a_3B_2\right)}, \qquad (3.63)$$

and

$$A=B=-\frac{(2n+1)b_1}{4(n+1)b_2}, \qquad (3.64)$$

which prompts the same constraint relations as in the parabolic law case. Finally, the dark 1-soliton solution to the NLSE with dual-power law nonlinearity is given by

$$q(x,y,t) = \left[A\left\{1+\tanh\left(B_1x+B_2y-vt\right)\right\}\right]^{\frac{1}{2n}} e^{i(-\kappa_1 x-\kappa_2 y+\omega t+\theta)}. \qquad (3.65)$$

General text of this section.

3.3 SINGULAR SOLITONS

In this section we discuss the singular solitons of the NLSE with STD. Singular solitons can serve as a possible analytical model for the formation of rogue waves. However, this is not yet confirmed. It is not yet known if log law nonlinearity supports singular optical solitons. Therefore, we only consider the remaining four types of nonlinearities.

3.3.1 KERR LAW

For Kerr law nonlinearity, we write [23, 24]

$$P(x,y,t) = A\,\mathrm{csch}^p s \qquad (3.66)$$

In this case, just as for dark solitons, A, B_1 and B_2 are free parameters. Substituting into (52) gives

$$\begin{aligned} &\left\{\omega + a_1\left(\kappa_1^2+\kappa_2^2\right) - \omega\left(a_2\kappa_1+a_3\kappa_2\right)\right\}\mathrm{csch}^p s - bA^2\,\mathrm{csch}^{3p} s \\ + \; &\left\{a_1\left(B_1^2+B_2^2\right) - v\left(a_2B_1+a_3B_2\right)\right\}\left\{p^2\,\mathrm{csch}^p s - p(p+1)\,\mathrm{csch}^{p+2} s\right\} = 0. \end{aligned} \qquad (3.67)$$

Here, we find $p = 1$ and

$$q(x,y,t) = A \text{ csch}\left(B_1 x + B_2 y - vt\right) e^{i(-\kappa_1 x - \kappa_2 y + \omega t + \theta)} \tag{3.68}$$

with all the parameters and constraint conditions defined through (3.8)-(3.10).

3.3.2 POWER LAW

For power law nonlinearity, substituting this hypothesis (3.66) into (3.12) leads to

$$\begin{aligned} & \left\{\omega + a_1\left(\kappa_1^2 + \kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2\right)\right\} \operatorname{csch}^p s - bA^{2n} \operatorname{csch}^{(2n+1)p} s \\ + \quad & \left\{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2 B_1 + a_3 B_2\right)\right\}\left\{p^2 \operatorname{csch}^p s - p(p+1)\operatorname{csch}^{p+2} s\right\} = 0. \end{aligned} \tag{3.69}$$

The singular 1-soliton solution is

$$q(x,y,t) = A \text{ csch}^{\frac{1}{n}}\left(B_1 x + B_2 y - vt\right) e^{i(-\kappa_1 x - \kappa_2 y + \omega t + \theta)} \tag{3.70}$$

where ω and A are defined by (3.15) and (3.16) respectively.

3.3.3 PARABOLIC LAW

Our starting point for the singular 1-soliton solution is the ansatz [23, 24]

$$P(x,y,t) = \frac{A}{(D_1 + \sinh s)^p} \tag{3.71}$$

where, D_1 is a parameter. Substituting this hypothesis (3.71) into (3.18) leads to

$$
\begin{aligned}
& \left[\omega + a_1\left(\kappa_1^2+\kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2\right)\right. \\
- & \left. p^2\left\{a_1\left(B_1^2+B_2^2\right) - v\left(a_2B_1+a_3B_2\right)\right\}\right] \\
+ & p(2p+1)D_1\frac{a_1\left(B_1^2+B_2^2\right) - v\left(a_2B_1+a_3B_2\right)}{D_1+\sinh s} \\
+ & p(p+1)\left(D_1^2-1\right)\frac{v\left(a_2B_1+a_3B_2\right) - a_1\left(B_1^2+B_2^2\right)}{(D_1+\sinh s)^2} \\
- & \frac{b_1A^2}{(D_1+\sinh s)^{2p}} - \frac{b_2A^4}{(D_1+\sinh s)^{4p}} = 0.
\end{aligned}
\tag{3.72}
$$

The same parameter definitions and constraint conditions as given by (3.21)-(3.25) are obtained. Therefore, the singular 1-soliton solution is

$$
q(x,y,t) = \frac{A}{\left\{D_1 + \sinh\left(B_1x + B_2y - vt\right)\right\}^{\frac{1}{2}}} e^{i(-\kappa_1 x - \kappa_2 y + \omega t + \theta)}. \tag{3.73}
$$

3.3.4 DUAL-POWER LAW

For dual-power law nonlinearity, we assume [23, 24]

$$
P(x,y,t) = \frac{A}{(D_2+\sinh s)^p} \tag{3.74}
$$

where D_2 is a constant to be determined. Substituting (3.74) into (3.26) leads to

$$
\begin{aligned}
& \left[\omega + a_1\left(\kappa_1^2 + \kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2\right)\right. \\
- & \left. p^2\left\{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2B_1 + a_3B_2\right)\right\}\right] \\
+ & \ p(2p+1)D_2\frac{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2B_1 + a_3B_2\right)}{D_2 + \sinh s} \\
+ & \ p(p+1)\left(D_2^2 - 1\right)\frac{v\left(a_2B_1 + a_3B_2\right) - a_1\left(B_1^2 + B_2^2\right)}{(D_2 + \sinh s)^2} \\
- & \ \frac{b_1A^{2n}}{(D_2 + \sinh s)^{2np}} - \frac{b_2A^{4n}}{(D_2 + \sinh s)^{4np}} = 0,
\end{aligned}
\tag{3.75}
$$

and also leads to the same parameter relations expressed through equations (3.30)-(3.34). The singular 1-soliton solution to (34) is

$$
q(x,y,t) = \frac{A}{\left\{D_2 + \sinh\left(B_1x + B_2y - vt\right)\right\}^{\frac{1}{2n}}}e^{i(-\kappa_1x - \kappa_2y + \omega t + \theta)}. \tag{3.76}
$$

Chapter 4

SEMI-INVERSE VARIATIONAL PRINCIPLE

The third method of integrability that will be studied in this chapter is the so called semi-inverse variational principle (SVP). In an attempt to solve (1.1), we look for traveling wave hypothesis as given by (2.1). Upon substituting (2.1) into (1.1) leads to the velocity given by (2.5) and the ordinary differential equation (2.4). Multiplying both sides of (2.4) by g' and integrating leads to

$$\begin{aligned} & \left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}\left(g'\right)^2 \\ = & \left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}g^2-2\int^g hF\left(h^2\right)dh+K \end{aligned} \tag{4.1}$$

where K is the integration constant. The stationary integral is then defined as [3, 26]

$$\begin{aligned} J &= \int_{-\infty}^{\infty} K ds \\ &= \int_{-\infty}^{\infty}\left[\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}\left(g'\right)^2\right. \\ &- \left.\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2\right)\right\}g^2+2\int^g hF\left(h^2\right)dh\right]ds \end{aligned} \tag{4.2}$$

Finally, underlying the SVP is to postulate a trial solution for the 1-soliton solution. Hence we assume

$$g(s)=Af\left[\operatorname{sech}(Bs)\right], \tag{4.3}$$

or

$$g(s) = Ae^{-B^2 s^2} \tag{4.4}$$

depending on the type of nonlinearity that is being considered. The functional f in (4.3) is based on one of the first four forms of nonlinearity. The second form of assumption is for the log law nonlinearity. Here, A is the amplitude and B is the inverse width of the soliton or Gausson. The SVP states that the amplitude A and the inverse width B can be obtained from the following coupled system of equations [3, 26]

$$\frac{\partial J}{\partial A} = 0, \tag{4.5}$$

and

$$\frac{\partial J}{\partial B} = 0. \tag{4.6}$$

This principle will now be applied to the NLSE and studied in details in the following five subsections.

4.1 KERR LAW

For Kerr law nonlinearity, the stationary integral is given by [3]

$$\begin{aligned} J &= \int_{-\infty}^{\infty} \left[\left\{ a_1 \left(B_1^2 + B_2^2 \right) - v \left(a_2 B_1 + a_3 B_2 \right) \right\} \left(g' \right)^2 \right. \\ &- \left. \left\{ a_1 \left(\kappa_1^2 + \kappa_2^2 \right) - \omega \left(a_2 \kappa_1 + a_3 \kappa_2 - 1 \right) \right\} g^2 + \frac{b g^4}{2} \right] ds. \end{aligned} \tag{4.7}$$

The assumption for $g(s)$ in this case is [3, 11]

$$g(s) = A\,\mathrm{sech}(Bs). \tag{4.8}$$

Upon substituting this into the stationary integral and simplifying leads to

$$\begin{aligned} J &= \frac{2A^2}{3B}\left[\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}B^2\right. \\ &- \left. 3\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}+bA^2\right]. \end{aligned} \tag{4.9}$$

Setting the partial derivatives of J with respect to A and B to zero results in

$$\begin{aligned} &\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}B^2 \\ &- 3\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}+2bA^2=0, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} &\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}B^2 \\ &+ 3\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}-bA^2=0. \end{aligned} \tag{4.11}$$

Solving the coupled system (4.10) and (4.11) gives

$$A = \left[\frac{2\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}}{b}\right]^{\frac{1}{2}}, \tag{4.12}$$

and

$$B = \left[\frac{\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)-a_1\left(\kappa_1^2+\kappa_2^2\right)}{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)}\right]^{\frac{1}{2}}, \tag{4.13}$$

while the amplitude-width relationship is

$$B = A\left[\frac{b}{2\left\{v\left(a_2B_1 + a_3B_2\right) - a_1\left(B_1^2 + B_2^2\right)\right\}}\right]^{\frac{1}{2}}. \tag{4.14}$$

Requiring that $A > 0$ and $B > 0$ means

$$b\left\{a_1\left(\kappa_1^2 + \kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2 - 1\right)\right\} > 0, \tag{4.15}$$

$$\left\{\omega\left(a_2\kappa_1 + a_3\kappa_2 - 1\right) - a_1\left(\kappa_1^2 + \kappa_2^2\right)\right\}\left\{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2B_1 + a_3B_2\right)\right\} > 0, \tag{4.16}$$

and

$$b\left\{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2B_1 + a_3B_2\right)\right\} < 0. \tag{4.17}$$

The 1-soliton solution to (2.8) is (2.15) where the parameters are defined in (4.12)-(4.14) and the constraints are as in (4.15)-(4.17).

4.2 POWER LAW

For power law nonlinearity, the stationary integral is

$$\begin{aligned} J &= \int_{-\infty}^{\infty}\Big[\left\{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2B_1 + a_3B_2\right)\right\}\left(g'\right)^2 \\ &- \left\{a_1\left(\kappa_1^2 + \kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2 - 1\right)\right\}g^2 + \frac{bg^{2n+2}}{n+1}\Big]\, ds. \end{aligned} \tag{4.18}$$

Here, soliton profile is taken to be [3, 11]

$$g(s) = A\text{sech}^{\frac{1}{n}}(Bs). \tag{4.19}$$

Here, soliton profile is taken to be [3, 11]

$$g(s) = A\mathrm{sech}^{\frac{1}{n}}(Bs). \tag{4.19}$$

Substituting (4.19) anto (4.18), we get

$$\begin{aligned} J &= \frac{A^2}{n(n+1)(n+2)B}\left[\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}(n+1)B^2\right. \\ &- \left. n(n+1)(n+2)\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}+2nbA^{2n}\right]\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}, \end{aligned} \tag{4.20}$$

where $\Gamma(x)$ is the Gamma function. Using (4.5) and (4.6),

$$\begin{aligned} &\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}B^2 \\ -\ &n(n+2)\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}+2nbA^2=0, \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} &(n+1)\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}B^2 \\ +\ &n(n+1)(n+2)\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}-2nbA^{2n}=0, \end{aligned} \tag{4.22}$$

from which we deduce that

$$A = \left[\frac{(n+1)\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}}{b}\right]^{\frac{1}{2n}}, \tag{4.23}$$

and

$$B = n\left[\frac{\omega\left(a_2\kappa_1 + a_3\kappa_2 - 1\right) - a_1\left(\kappa_1^2 + \kappa_2^2\right)}{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2B_1 + a_3B_2\right)}\right]^{\frac{1}{2}}, \tag{4.24}$$

while the amplitude-width relationship is given by

$$B = nA^n\left[\frac{b}{(n+1)\left\{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2B_1 + a_3B_2\right)\right\}}\right]^{\frac{1}{2}}. \tag{4.25}$$

4.3 PARABOLIC LAW

Here, the stationary integral is expressed as

$$\begin{aligned} J &= \int_{-\infty}^{\infty}\Big[\left\{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2B_1 + a_3B_2\right)\right\}\left(g'\right)^2 \\ &- \left\{a_1\left(\kappa_1^2 + \kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2 - 1\right)\right\}g^2 + \frac{b_1g^4}{2} + \frac{b_2g^6}{3}\Big]\,ds. \end{aligned} \tag{4.26}$$

The trial function for $g(s)$ is [11]

$$g(s) = \frac{A}{\left\{D_1 + \cosh(Bs)\right\}^{\frac{1}{2}}} \tag{4.27}$$

for some external parameter D_1. Substituting this into the stationary integral (4.26)

$$\begin{aligned} J &= \frac{2}{3}\left\{a_1\left(B_1^2 + B_2^2\right) - v\left(a_2B_1 + a_3B_2\right)\right\}A^2BM_1 \\ &- 2\left\{a_1\left(\kappa_1^2 + \kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2 - 1\right)\right\}\frac{A^2M_2}{B} \\ &+ \frac{b_1M_3A^4}{3B} + \frac{4b_2M_4A^6}{45B}, \end{aligned} \tag{4.28}$$

where M_j $(1 \leq j \leq 4)$ is

$$M_1 = F\left(3, 1, -\frac{1}{2}; \frac{1-D_1}{2}\right), \tag{4.29}$$

$$M_2 = F\left(1, 1, \frac{3}{2}; \frac{1-D_1}{2}\right), \tag{4.30}$$

$$M_3 = F\left(2, 2, \frac{5}{2}; \frac{1-D_1}{2}\right), \tag{4.31}$$

and

$$M_4 = F\left(3, 3, \frac{7}{2}; \frac{1-D_1}{2}\right). \tag{4.32}$$

Here, $F(\alpha, \beta; \gamma; z)$ is the Gauss' hypergeometric function that is defined as [10]

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)} \frac{z^n}{n!}. \tag{4.33}$$

The convergence criteria for the hypergeometric function is given by

$$\gamma < \alpha + \beta, \tag{4.34}$$

and

$$|z| < 1 \tag{4.35}$$

the latter of which, implies

$$-1 < D_1 < 3. \tag{4.36}$$

Now, equations (4.5) and (4.6) respectively produce

$$\begin{aligned} & 5\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}B^2M_1 \\ - \; & 15\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}M_2+5b_1A^2M_3+2b_2A^4M_4=0, \end{aligned} \tag{4.37}$$

and

$$\begin{aligned} & 5\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}B^2M_1 \\ - \; & 15\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}M_2+5b_1A^2M_3+2b_2A^4M_4=0. \end{aligned} \tag{4.38}$$

Solving for the amplitide A and the inverse width B

$$A=\left[\frac{3Y-45b_1M_3}{32b_2M_4}\right]^{\frac{1}{2}}, \tag{4.39}$$

where

$$Y=\sqrt{225b_1^2M_3^2+1280b_2M_2M_4\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}}.$$

$$B=A\left[\frac{15b_1M_3+8b_2M_4A^2}{60M_1\left\{v\left(a_2B_1+a_3B_2\right)-a_1\left(B_1^2+B_2^2\right)\right\}}\right]^{\frac{1}{2}}. \tag{4.40}$$

These imply the constraint conditions

$$256b_2M_2M_4\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}+45b_1^2M_3^2>0, \tag{4.41}$$

and

$$M_1\left(15b_1M_3+8b_2M_4A^2\right)\left\{v\left(a_2B_1+a_3B_2\right)-a_1\left(B_1^2+B_2^2\right)\right\}>0. \tag{4.42}$$

4.4 DUAL-POWER LAW

The stationary integral now is

$$\begin{aligned} J &= \int_{-\infty}^{\infty}\left[\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}(g')^2\right. \\ &- \left.\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}g^2+\frac{b_1g^{2n+2}}{n+1}+\frac{b_2g^{4n+2}}{2n+1}\right]ds. \end{aligned} \tag{4.43}$$

The trial function is [11]

$$g(s)=\frac{A}{\{D_2+\cosh(Bs)\}^{\frac{1}{2n}}}, \tag{4.44}$$

where D_2 is a parameter. On substituting (4.24) into (4.23)

$$\begin{aligned} J &= \frac{2}{2^{\frac{1}{n}}}\left[\frac{n}{n+2}\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}A^2BN_1\right. \\ &- \left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}\frac{A^2N_2}{B} \\ &+ \left.\frac{b_1A^{2n+2}N_3}{(n+1)B}+\frac{(n+1)b_2A^{4n+2}N_4}{(n+2)(2n+1)(3n+2)B}\right], \end{aligned} \tag{4.45}$$

where N_j, $(1 \leq j \leq 4)$

$$N_1 = F\left(\frac{1}{n}+2, \frac{1}{n}, -\frac{1}{2}; \frac{1-D_2}{2}\right) \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}, \tag{4.46}$$

$$N_2 = F\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}+\frac{1}{2}; \frac{1-D_2}{2}\right) \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}, \tag{4.47}$$

$$N_3 = F\left(\frac{1}{n}+1, \frac{1}{n}+1, \frac{1}{n}+\frac{3}{2}; \frac{1-D_2}{2}\right) \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}, \tag{4.48}$$

and

$$N_4 = F\left(\frac{1}{n}+2, \frac{1}{n}+2, \frac{1}{n}+\frac{5}{2}; \frac{1-D_2}{2}\right) \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}. \tag{4.49}$$

The existence criteria is

$$-1 < D_2 < 3. \tag{4.50}$$

Now, equations (4.5) and (4.6) respectively reduce to

$$\begin{aligned}
& n(3n+2)\left\{a_1\left(B_1^2+B_2^2\right) - v\left(a_2B_1+a_3B_2\right)\right\} N_1B^2 \\
- \; & (n+2)(3n+2)\left\{a_1\left(\kappa_1^2+\kappa_2^2\right) - \omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\} N_2 \\
+ \; & (n+2)(3n+2)b_1N_3A^{2n} + (n+1)b_2N_4A^{4n} = 0,
\end{aligned} \tag{4.51}$$

and

$$
\begin{aligned}
& n(n+1)(2n+1)(3n+2)\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}N_1B^2 \\
+ \quad & (n+1)(n+2)(2n+1)(3n+2)\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}N_2 \\
- \quad & (n+2)(2n+1)(3n+2)b_1N_3A^{2n}-(n+1)^2N_4A^{4n}=0. \qquad (4.52)
\end{aligned}
$$

Solving for the amplitude A

$$
A=\left[\frac{Z}{4n(n+1)b_2N_4\sqrt{(n+2)(3n+2)}}-\frac{b_1(2n+1)N_3}{4b_2(n+1)N_4}\right]^{\frac{1}{2n}}, \qquad (4.53)
$$

where

$$
Z=\sqrt{n^2(n+2)(2n+1)^2(3n+2)b_1^2N_3^2-16n(n+1)^3(2n+1)b_2N_2N_4\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\}}.
$$

The connection between the amplitude A and the parameter B is

$$
B=A^n\left[\frac{(n+2)(2n+1)(3n+2)b_1N_3+2(n+1)^2b_2N_4A^{2n}}{2(n+1)(2n+1)(3n+2)N_1\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2B_1+a_3B_2\right)\right\}}\right]^{\frac{1}{2}}. \qquad (4.54)
$$

Finally, the constraints are

$$
\begin{aligned}
& 16(n+1)^3b_2N_2N_4\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)\right\} \\
- \quad & n(n+2)(2n+1)(3n+2)b_1^2N_3^2<0, \qquad (4.55)
\end{aligned}
$$

and

$$\left\{(n+2)(2n+1)(3n+2)b_1 N_3 + 2(n+1)^2 b_2 N_4 A^{2n}\right\}$$
$$\left\{a_1\left(B_1^2+B_2^2\right) - v\left(a_2 B_1 + a_3 B_2\right)\right\} N_1 > 0. \tag{4.56}$$

4.5 LOG LAW

Finally, for log law nonlinearity,

$$\begin{aligned} J &= \int_{-\infty}^{\infty} \Big[\left\{a_1\left(B_1^2+B_2^2\right) - v\left(a_2 B_1 + a_3 B_2\right)\right\}(g')^2 \\ &- \left\{a_1\left(\kappa_1^2+\kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2 - 1\right)\right\} g^2 + 2bg^2 \ln g - bg^2\Big]\, ds. \end{aligned} \tag{4.57}$$

The assumption for $g(s)$ in this case is given by (4.4) [3]. Upon inserting this expression into the stationary integral (4.57) leads to

$$\begin{aligned} J &= \frac{\sqrt{2\pi}}{4B}\Big[2\left\{a_1\left(B_1^2+B_2^2\right) - v\left(a_2 B_1 + a_3 B_2\right)\right\} A^2 B^2 \\ &- 2\left\{a_1\left(\kappa_1^2+\kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2 - 1\right)\right\} A^2 + A^2\left(3b + 4b\ln A\right)\Big]. \end{aligned} \tag{4.58}$$

By (4.5) and (4.6)

$$\begin{aligned} & 2\left\{a_1\left(B_1^2+B_2^2\right) - v\left(a_2 B_1 + a_3 B_2\right)\right\} B^2 \\ - & 2\left\{a_1\left(\kappa_1^2+\kappa_2^2\right) - \omega\left(a_2\kappa_1 + a_3\kappa_2 - 1\right)\right\} + 3b + 4blnA + 2b = 0, \end{aligned} \tag{4.59}$$

and

$$
\begin{aligned}
& 2\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2 B_1+a_3 B_2\right)\right\} B^2 \\
+ \quad & 2\left\{a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2 \kappa_1+a_3 \kappa_2-1\right)\right\}+3b-4b\ln A=0.
\end{aligned}
\tag{4.60}
$$

Solving the coupled system (4.59)-(4.60) gives the amplitude of the Gausson as

$$
A=\exp\left\{\frac{b+a_1\left(\kappa_1^2+\kappa_2^2\right)-\omega\left(a_2\kappa_1+a_3\kappa_2-1\right)}{2b}\right\}, \tag{4.61}
$$

and the parameter B is

$$
B=\sqrt{\frac{b}{2\left\{v\left(a_2 B_1+a_3 B_2\right)-a_1\left(B_1^2+B_2^2\right)\right\}}}, \tag{4.62}
$$

which prompts the constraint condition

$$
b\left\{a_1\left(B_1^2+B_2^2\right)-v\left(a_2 B_1+a_3 B_2\right)\right\}<0. \tag{4.63}
$$

Chapter 5

CONCLUSIONS

Three different approaches were used to obtain the 1-soliton solution to the (2+1)-dimensional NLSE in presence of STD. These include the first integral method, ansatz method and the SVP. The application of these methods provide the bright, dark as well as singular soliton solutions for five types of nonlinearity. They are Kerr Law, Power Law, Parabolic Law, Dual-Power Law and the Log Law nonlinearity. It is for the log law nonlinearity that only bright soliton solutions are possible.

In future, there are several related problems that can be tackled. They include obtaining the quasi-stationary solitons in the presence of perturbation terms, developing the quasi-particle theory for suppressing the soliton-soliton interaction. The study of soliton solutions for vector NLSE in (2+1)-dimensions is also pending. The variational approach will also be applied, with perturbation terms, in an attempt to obtain the soliton solution as well as soliton radiation. Several other approaches will be implemented in order to extract solitons and Gaussons.

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RESEARCH INTERESTS

Optical Solitons, Variational Principles.

PUBLICATIONS:

1: **Yanan Xu**, Zlatko Jovanoski, Abdelaziz Bouasla, Houria Triki, Luminita Moraru and Anjan Biswas, Optical Solitons in Multi-Dimensions with Spatio-Temporal Dispersion and Non-Kerr Law Nonlinearity, Journal of Nonlinear Optical Physics and Materials, 22(3), 30 pages, 2013.

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