EXACT TOPOLOGICAL SOLITON SOLUTIONS OF THE DISPERSIVE, STRONGLY PERTURBED, AND 2D SINE-GORDON TYPE EQUATIONS

by

STEPHEN JOHNSON

A DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in the
Mathematical Physics Graduate Program
of Delaware State University

DOVER, DELAWARE
August 2018

This dissertation is approved by the following members of the Final Oral Review Committee:

Dr. Sokratis Makrogiannis, Committee Chairperson, Department of Mathematical Sciences, Delaware State University
Dr. Essaid Zerrad, Committee Member, Department of Physics and Engineering, Delaware State University
Dr. Matthew Tanzy, Committee Member, Department of Mathematical Sciences, Delaware State University
Dr. Yuriy Markushin, Committee Member, Department of Physics and Engineering, Delaware State University
Dr. Stephen Barr, External Committee Member, Department of Physics and Astronomy, University of Delaware

DEDICATION

This dissertation is dedicated to my loving wife Megan. Without her support I would not have finished this long journey.

ACKNOWLEDGEMENTS

I would like to thank my advisors, Dr. Makrogiannis and Dr. Zerrad, for helping, guiding, and supporting me through this process. I encountered many difficulties along the way, and they were there for me when I needed them most. Thank you. I would also like to thank my committee members, Dr. Tanzy, Dr. Markushin, and Dr. Barr, for their insightful questions, comments, and suggestions that enriched my research efforts.

ABSTRACT

This dissertation uses the Ansatz method to solve for exact topological soliton solutions to sine-Gordon type equations. Single, double, and triple sine-Gordon and sine-cosine-Gordon equations are investigated along with dispersive and highly dispersive variations. After these solutions are found, strong perturbations are added to each equation and the new solutions are found. In solving both the perturbed and unperturbed sine-Gordon type equations, constraints are imposed on the parameters. After finding exact solutions to the dispersive sine-Gordon type equations, three new solutions to the 2D sine-Gordon equation are found. These solutions include the domain wall, the breather, and the domain wall collision. Of particular interest is the Domain wall collision to the 2D sine-Gordon equation, which to the authors' knowledge had not previously been presented in the literature.

The first chapter will begin by giving the historical context of solitons and the sine-Gordon equation. It will be shown here that the results found in the later chapters will be important to the study of Josephson junctions, crystal dislocations, ultra-short optical pulses, relativistic field theory, and elementary particles. This chapter will continue on to show the derivation of the discrete sine-Gordon equation by means of the Hamiltonian. The continuous sine-Gordon equation will be shown to arise from the discrete version in the long wave limit. This approximation will also show how the higher order dispersion terms arise. This chapter will conclude by explaining the Ansatz method in detail. It will also show some other common methods for the study of solitons.

The second, third, and fourth chapters will find exact solutions to the strongly perturbed sine-Gordon type equations using the Ansatz method. The sine-Gordon type equations studied will include the single, double, and triple sine-Gordon equations, and all of their

sine-cosine-Gordon analogs. In addition to these equations, higher order dispersive versions will also be studied. These will include both fourth and sixth order dispersion.

The fifth chapter will find exact solutions to the 2D sine-Gordon equation. This study will also be carried out using the Ansatz method. However, in this case special relativity will be used to turn a stationary solution into a moving solution. This will be performed in the analog sense to how it is commonly done for the 1D sine-Gordon equation.

The sixth chapter will summarize the dissertation and give some final remarks.

TABLE OF CONTENTS

LIST OF FIGURES AND ILLUSTRATIONS

Chapter I: INTRODUCTION

1.1 HISTORICAL CONTEXT

This dissertation falls within the larger study of solitons and solitary waves. As such, the reader may be aided by the following definitions. A solitary wave is a localized wave that propagates without distortion of its size or shape. A soliton is a special solitary wave that may interact or collide with other solitons and emerge unchanged except for possibly a phase shift. Solitons are formed by nonlinear partial differential equations when the nonlinear forces, which cause a steepening effect, are exactly balanced by the dispersive forces, which cause a flattening effect [12].

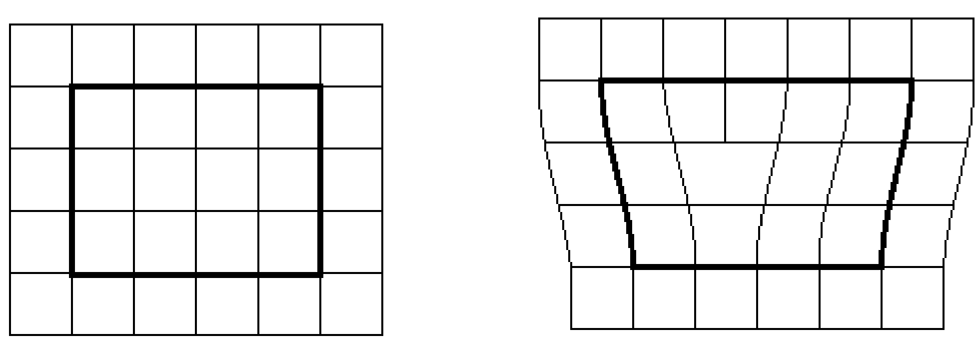
A topological soliton (TS) is a special type of soliton that can only occur when a particular medium allows for degenerate ground states. A TS is therefore a soliton whose boundary points end in topologically different ground states. For any TS, the ground state boundary points are fixed and can never jump to a different ground state. Therefore one TS may never transform into a TS with a different topology having different endpoints. A double soliton in the context of this dissertation is a TS with twice the height of a single TS, where a single TS has endpoints on adjacent ground states. A triple soliton has three times the height of a single TS. A kink is another term for a TS [8].

The sine-Gordon equation (SGE) has applications to Josephson junctions, crystal dislocations, ultra-short optical pulses, relativistic field theory, and elementary particles [1].

A Josephson junction is a pair of superconductors separated by a thin material that is not superconducting. Brian Josephson predicted that a pair of superconducting electrons could tunnel through the non-superconducting material [30]. Certain materials, called superconductors, have a special property that when they are cooled below a critical temperature, the electrical resistance drops to zero and the magnetic fields are expelled from within the material. Because of this, an electric current can maintain itself indefinitely without a power source.

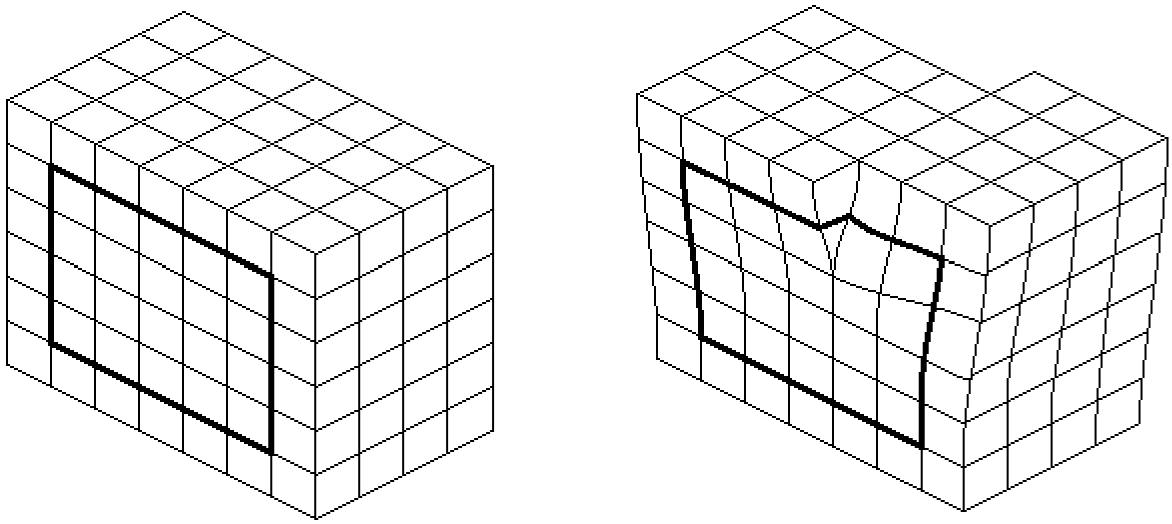
In a Josephson junction, a direct supercurrent exists across the weak link between the two superconductors in the absence of an applied voltage. A supercurrent is a current within a superconductor, or a current that does not dissipate. If a voltage is applied across this barrier, an alternating current will develop [2].

Josephson junctions are found in computer circuitry, increasing the speed of computations. They are also found in Superconducting Quantum Interfering Devices (SQUIDs). SQUIDs can measure minute changes in voltages and magnetic fields. Because of this, SQUIDs can be used to measure neural activity in the brain, heart activity [21], and even submarine detection [23].



(a) Standard lattice (b) Inserted half-plane

Figure 1.1: Edge Dislocation



(a) Standard Lattice

(b) Slipped half-plane

Figure 1.2: Screw Dislocation

A crystal dislocation is an irregularity within the crystal structure. There are two types of dislocations. An edge dislocation occurs when one plane of atoms only extends half-way through the crystal. This causes the planes to bend around it. A screw dislocation looks like the Riemann surface of the complex logarithm. Dislocations can also be a combination of the two types. Studying these dislocations can help material scientists improve the strength of metals. The kink solitons of the SGE can be used to model the interactions of these dislocations [19].

Ultrashort pulses allow scientists to study ultrashort processes and allow for optical data transmission. Some ultrashort processes of interest include electron dynamics within semiconductors, light-induced phase changes of metals, plasma dynamics, and chemical reactions [37]. Light pulses can be used to transmit data at very high rates due to high optical frequencies [39].

In relativistic field theory, kinks can be used to model undiscovered particles such as magnetic monopoles and cosmic strings [36].

Recently, researchers have begun studying the sine-cosine-Gordon equation (SCGE). One of the first such studies was done by Wazwaz in 2006 [43]. This work was later expanded on by Kuo and Hu in 2009 [33] to show how the SCGE can be used to model a spin 1/2 chain. Throughout this dissertation solutions to the sine-cosine-Gordon type equations will be found alongside their SG type equation analogs.

In all of the above applications, the SGE is not enough to sufficiently model each situation. Instead there are other terms beyond the normal SGE that are required for a more accurate description of the underlying physics. Perturbation terms are often times added to the SGE to represent these extra qualities. Weak perturbation analysis is used when the extra terms are very small in comparison to the regular SGE terms [18, 28]. However, since the physics demands these terms are of the same order, they must be called strong perturbations [13, 15, 20].

The 2D SGE has applications to ferromagnetic media [34] and light bullets. Typical pulses of light consist of an envelope containing hundreds or even thousands of internal oscillations. The usual way to study these light pulses is by the Nonlinear Schrdinger Equation (NLSE), which approximates the slowly varying envelope. Light bullets are ultra-short optical pulses that have only a few oscillations within the wave envelope. This means the oscillations are on the same order of magnitude as the envelope itself. Because of this, the NLSE no longer accurately approximates the wave dynamics. It was found that the 2D SGE is a better approximation for this phenomenon [44].

This dissertation studies three new solutions to the 2D SGE. These solutions all have immediate 1D analogs to the 1D SGE. The domain wall is the 2D generalization of the kink. The domain wall collision is the 2D generalization of soliton collisions of the 1D SGE. Breathers of the SGE are also generalized to their 2D counterparts.

The goal of this dissertation is to find exact solutions to strongly perturbed sine-Gordon (SG) type equations and the 2D SGE. This will be accomplished by means of the Ansatz method.

1.2 SINE GORDON EQUATION DERIVATION

The discrete SGE describes a harmonically coupled chain of atoms in a periodic potential. The Hamiltonian of this system is given by [42]

$$\mathcal{H} = \sum_n \frac{1}{2}\dot{q}_n^2 + \frac{1}{2\Delta x^2}(q_{n+1} - q_n)^2 + a(1 - \cos q_n)$$

where each atom is coupled to its nearest neighbors by linear springs. Each atom also experiences a local potential with energy $a(1 - \cos q)$. The Hamiltonian equations are of course

$$\frac{dq_n}{dt} = \frac{\partial \mathcal{H}}{\partial p_n} \qquad \text{and} \qquad \frac{dp_n}{dt} = -\frac{\partial \mathcal{H}}{\partial q_n}$$

where the momentum p_n in this case is just $\dot{q}_n$. Solving the Hamiltonian equations leads to the discrete SGE, given by [26]

$$\ddot{q}_n - \frac{1}{\Delta x^2}\left(q_{n+1} - 2q_n + q_{n-1}\right) + a \sin q_n = 0$$

where q_n represents the displacement of the n^{th} atom and Δx is the lattice spacing parameter. The second-order difference can be approximated by

$$\frac{1}{\Delta x^2}\left(q_{n+1} - 2q_n + q_{n-1}\right) = 2\sum_{m=0}^{\infty} \frac{\Delta x^{2m}}{(2m+2)!}\partial_{xx}^m q_{xx}(n\Delta x)$$

The first few terms of the Taylor expansion are

$$\frac{1}{\Delta x^2}\left(q_{n+1} - 2q_n + q_{n-1}\right) = q_{xx}(n\Delta x) + \frac{\Delta x^2}{12} q_{xxxx}(n\Delta x) + \frac{\Delta x^4}{360} q_{xxxxxx}(n\Delta x) + \cdots$$

The continuous SGE is thus the first-order approximation to the discrete SGE representing a crystal lattice. The dispersive SGE is the second-order approximation, and the highly dispersive SGE is the third-order approximation.

In the same way that the continuous SGE was derived from the above Hamiltonian, slightly modified versions lead to the double and triple SG equations.

$$\mathcal{H} = \sum_n \frac{1}{2}\dot{q}_n^2 + \frac{1}{2\Delta x^2}(q_{n+1} - q_n)^2 + a_1(1 - \cos q_n) + \frac{a_2}{2}(1 - \cos 2q_n)$$

This Hamiltonian, with the addition of the $(a_2/2)(1 - \cos 2q_n)$ term leads to the double SGE

$$q_{tt} - q_{xx} + a_1 \sin q + a_2 \sin 2q = 0$$

and the addition of the final term $(a_3/3)(1 - \cos 3q_n)$

$$\mathcal{H} = \sum_n \frac{1}{2}\dot{q}_n^2 + \frac{1}{2\Delta x^2}(q_{n+1} - q_n)^2 + a_1(1 - \cos q_n) + \frac{a_2}{2}(1 - \cos 2q_n) + \frac{a_3}{3}(1 - \cos 3q_n)$$

leads to the triple SGE

$$q_{tt} - q_{xx} + a_1 \sin q + a_2 \sin 2q + a_3 \sin 3q = 0$$

1.3 DESCRIPTION OF THE ANSATZ METHOD

There are many methods of finding solutions to nonlinear partial differential equations. Some of the more popular methods include Lie symmetry analysis [7, 40, 41], the exp-function method [4, 13, 14, 15, 16, 40], the tanh method [27], the G'/G method [4, 13, 15, 16, 40], the F-expansion method [3, 16], the mapping method [4, 7, 14], semi-inverse variational principle [6, 4], traveling waves [5, 13, 14, 16, 27, 41], and the Ansatz method [3, 7, 15, 20, 24, 38, 40]. This dissertation uses the Ansatz method to find exact solutions to SG type equations.

The typical nonlinear wave equation that will be studied in this dissertation is of the form

$$q_{tt} - q_{xx} + L(q) + N(q) = 0 \tag{1.1}$$

where x and t represent partial derivatives with respect to space and time respectively, and where $L(q)$ represents the linear terms, including derivatives of q, and $N(q)$ represents the nonlinear terms, including derivatives of q. The Ansatz method requires a guess for the solution. Call this guess the particular solution $\hat{q}(x,t)$. Once an initial guess is made for $\hat{q}(x,t)$, that solution is put into the original nonlinear wave equation (1.1)

$$\hat{q}_{tt} - \hat{q}_{xx} + L(\hat{q}) + N(\hat{q}) = 0$$

The above equation is then simplified down to a sum of linearly independent functions,

$$0 = \sum_{i=0}^{n} c_i F_i(x,t) \tag{1.2}$$

where c_i are the constant coefficients of the linearly independent functions $F_i(x,t)$. In order to satisfy (1.2), it is necessary to set all coefficients equal to zero

$$c_i = 0 \quad \forall i \in \{0, 1, 2, \ldots, n\}$$

In completing this step, one of two things will happen. If a critical parameter must be set to zero, then the particular solution $\hat{q}(x,t)$ has been proven to be invalid. Otherwise, this is proof that $\hat{q}(x,t)$ is a particular solution to (1.1). In the latter case, setting the c_i to zero will lead to some constraints on the parameters of (1.1) and the internal parameters of $\hat{q}(x,t)$.

Chapter II: TOPOLOGICAL SOLITONS DUE TO THE SINE-GORDON EQUATION AND ITS TYPE

This first section of this chapter will study the SGE and other similar equations without higher order dispersion terms. The same equations with the addition of fourth-order and sixth-order dispersion will be studied in chapters three and four, respectively.

2.1 SINE-GORDON EQUATION

The SGE studied in this chapter is

$$q_{tt} - k^2 q_{xx} + a \sin q = 0 \tag{2.1}$$

The kink Ansatz that will be used to solve this equation, given by [20], is

$$q(x,t) = 4 \arctan \left\{ A \exp \left[B(x - vt) \right] \right\} \tag{2.2}$$

The variable E will be used henceforth, where $E = A \exp[B(x - vt)]$. Inserting (2.2) into (2.1) yields

$$4 \frac{E - E^3}{\left(1 + E^2\right)^2} \left[B^2 \left(v^2 - k^2 \right) + a \right] = 0$$

Solving for B gives

$$B = \pm \sqrt{\frac{a}{k^2 - v^2}}$$

It turns out that $A = \pm e^{-Bx_0}$ represents the starting location of the soliton at x_0. Positive A represents a bright soliton whereas negative A represents a dark soliton. The sign of $A \cdot B$ determines the direction of the internal twist in the kink. These things hold true whenever the solution structure is an arctangent of an exponential.

The bright solution to the unperturbed SGE is

$$q(x,t) = 4\arctan\left\{\exp\left[\pm\sqrt{\frac{a}{k^2-v^2}}(x-x_0-vt)\right]\right\}$$

The result above is already well known [1, 8, 18, 20, 26, 28, 31]. Now that we have the exact solution to the unperturbed SGE, we will use the same Ansatz to find the exact solution to the strongly perturbed SGE. The variable R will be used to hold all of the perturbation terms, where

$$R = \beta q_t + \gamma q_x + \delta q_{xt} + \lambda q_{tt} + \sigma q_{xxt} + \nu q_{xxxx} \tag{2.3}$$

In Josephson junctions, β represents the dissipative losses of electrons tunneling across a dielectric barrier, γ comes about from an inhomogeneous part of the local inductance, δ accounts for the diffusion, λ results from an inhomogeneity of the capacitance, σ arises due to current losses along the barrier, and ν contains the higher order spatial dispersion [31].

The perturbed SGE is thus

$$q_{tt} - k^2 q_{xx} + a\sin q = R$$

Using the same Ansatz from (2.2), we get the governing equation

$$\begin{aligned} 0 = &\left(E - E^3\right)\left(1+E^2\right)^2\left[B^2\left(v^2 - \lambda v^2 + \delta v - k^2\right) + a\right] + E\left(1+E^2\right)^3\left[B\left(\beta v - \gamma\right)\right] \\ &+ \left(E - 6E^3 + E^5\right)\left(1+E^2\right)\left[B^3\sigma v\right] - \left(E - 23E^3 + 23E^5 - E^7\right)\left[B^4\nu\right] \end{aligned} \tag{2.4}$$

which has four linearly independent functions E, E^3, E^5, and E^7. Each must have its coefficient set to zero. Let $c_1 = B(\beta v - \gamma)$, $c_2 = B^2(v^2 - \lambda v^2 + \delta v - k^2) + a$, $c_3 = B^3\sigma v$,

$c_4 = -B^4\nu$. Then solving (2.4) is equivalent to solving the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -5 & -23 \\ 3 & -1 & -5 & 23 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.5}$$

Since this matrix has full rank, the only solution is the trivial solution. Setting the coefficients to zero gives the following results:

$$\begin{aligned} \gamma &= \beta v \\ \sigma &= 0 \\ \nu &= 0 \end{aligned} \tag{2.6}$$

The final result of the matrix equation (2.5) is

$$B = \pm\sqrt{\frac{a}{k^2 - \delta v - (1-\lambda)v^2}}$$

The bright soliton solution to the perturbed SGE is

$$q(x,t) = 4\arctan\left\{\exp\left[\pm\sqrt{\frac{a}{k^2 - \delta v - (1-\lambda)v^2}}(x - x_0 - vt)\right]\right\}$$

with the constraints found in (2.6). These results have already been reported in [20], and are reproduced here in order to show the method that will be used henceforth.

2.2 SINE-COSINE-GORDON EQUATION

The SCGE is

$$q_{tt} - k^2 q_{xx} + a\sin q + b\cos q = 0 \tag{2.7}$$

The kink Ansatz for this equation is

$$q(x,t) = 4\arctan\left\{A\exp\left[B\left(x - vt\right)\right]\right\} + C \tag{2.8}$$

Inserting (2.8) into (2.7) gives

$$0 = 4\left[B^2\left(v^2 - k^2\right) + a\cos C - b\sin C\right]\left(E - E^3\right) + \left(a\sin C + b\cos C\right)\left(1 - 6E^2 + E^4\right)$$

which gives the following two relationships:

$$C = -\arctan\left(\frac{b}{a}\right)$$
$$B = \pm\frac{\left(a^2 + b^2\right)^{\frac{1}{4}}}{\sqrt{k^2 - v^2}}$$

The solution to the unperturbed SCGE is

$$q(x,t) = 4\arctan\left\{\exp\left[\pm\frac{\left(a^2 + b^2\right)^{\frac{1}{4}}}{\sqrt{k^2 - v^2}}(x - x_0 - vt)\right]\right\} - \arctan\left(\frac{b}{a}\right)$$

The perturbed SCGE is

$$q_{tt} - k^2 q_{xx} + a\sin q + b\cos q = R$$

where R again contains all of the perturbation terms from (2.3). Using the same Ansatz from (2.8), we get the governing equation

$$\begin{aligned} 0 = &\left(E - E^3\right)\left(1 + E^2\right)^2\left[B^2\left(v^2 - \lambda v^2 + \delta v - k^2\right) + a\cos C - b\sin C\right] \\ &+ E\left(1 + E^2\right)^3\left[B\left(\beta v - \gamma\right)\right] + \left(E - 6E^3 + E^5\right)\left(1 + E^2\right)\left[B^3\sigma v\right] \\ &- \left(E - 23E^3 + 23E^5 - E^7\right)\left[B^4\nu\right] + \left(1 - 6E^2 + E^4\right)\left(1 + E^2\right)^2\left[a\sin C + b\cos C\right] \end{aligned} \tag{2.9}$$

Setting the coefficients of the linearly independent functions in (2.9) to zero as was done in section 2.1 leads to the same matrix equation (2.5) with the extra condition that

$$C = -\arctan\left(\frac{b}{a}\right)$$

The conditions for this equation are the same as (2.6) with the addition of

$$B = \pm\frac{(a^2+b^2)^{\frac{1}{4}}}{\sqrt{k^2-\delta v-(1-\lambda)v^2}}$$

The solution to the perturbed SCGE is

$$q(x,t) = 4\arctan\left\{\exp\left[\pm\frac{(a^2+b^2)^{\frac{1}{4}}}{\sqrt{k^2-\delta v-(1-\lambda)v^2}}(x-x_0-vt)\right]\right\} - \arctan\left(\frac{b}{a}\right)$$

2.3 DOUBLE SINE-GORDON EQUATION

The double sine-Gordon equation (DSGE) is

$$q_{tt} - k^2 q_{xx} + a_1 \sin q + a_2 \sin 2q = 0 \tag{2.10}$$

The first Ansatz we will investigate is

$$q(x,t) = 2\arctan\left\{A\sinh\left[B(x-vt)\right]\right\} \tag{2.11}$$

Substituting (2.11) into (2.10) gives the following equation

$$B^2\left(v^2-k^2\right)\left[\left(1-2A^2\right)S - A^2S^3\right] + a_1\left[S + A^2S^3\right] + 2a_2\left[S - A^2S^3\right] = 0$$

where $S = \sinh[B(x - vt)]$. Setting the coefficients of S and S^3 both to zero leads to the two constraints

$$A = \pm\sqrt{\frac{a_1}{a_1 - 2a_2}}$$
$$B = \pm\sqrt{\frac{2a_2 - a_1}{k^2 - v^2}}$$

The first solution to the unperturbed DSGE is

$$q(x,t) = 2\arctan\left\{\sqrt{\frac{a_1}{a_1 - 2a_2}}\sinh\left[\pm\sqrt{\frac{2a_2 - a_1}{k^2 - v^2}}(x - vt)\right]\right\}$$

For the soliton to exist, it is necessary that $2a_2 > a_1$ and $a_1 < 0$.

The second Ansatz for (2.10), found in [18], is

$$q(x,t) = 2\arctan\left\{A\tanh\left[B(x - vt)\right]\right\} \tag{2.12}$$

The resulting equation from combining (2.10) and (2.12) is

$$2B^2\left(1 + A^2\right)\left(k^2 - v^2\right)\left(T - T^3\right) + a_1\left(T + A^2T^3\right) + 2a_2\left(T - A^2T^3\right) = 0$$

where $T = \tanh[B(x - vt)]$. Once the coefficients of T and T^3 are set to zero, the resulting relationships are

$$A = \pm\sqrt{\frac{a_2 + a_1}{a_2 - a_1}}$$
$$B = \pm\frac{1}{2}\sqrt{\frac{a_2^2 - a_1^2}{a_2\left(k^2 - v^2\right)}}$$

The second solution to the unperturbed DSGE is

$$q(x,t) = 2\arctan\left\{\sqrt{\frac{a_2+a_1}{a_2-a_1}}\tanh\left[\pm\frac{1}{2}\sqrt{\frac{a_2^2-a_1^2}{a_2\left(k^2-v^2\right)}}(x-vt)\right]\right\}$$

where $a_2 > |a_1|$.

The perturbed DSGE is

$$q_{tt} - k^2 q_{xx} + a_1 \sin q + a_2 \sin 2q = R$$

Using the first Ansatz, (2.11), we get the equation

$$\begin{aligned}0 = {} & c_2\left[\left(1-2A^2\right)S + \left(1-4A^2\right)S^3 - \left(1+2A^2\right)S^5 - S^7\right] - a_1\left[S + 3S^3 + 3S^5 + S^7\right] \\ & + 2a_2\left[S + S^3 - S^5 - S^7\right] + c_1\left[1 + 3S^2 + 3S^4 + S^6\right]\cosh[B(x-vt)] \\ & + c_3\left[\left(1-2A^2\right) - \left(5-4A^2\right)S^2 - \left(5-6A^2\right)S^4 + S^6\right]\cosh[B(x-vt)] \\ & + c_4\left[\left(1-20A^2+24A^4\right)S + \left(1+56A^2-24A^4\right)S^3 + \left(21-20A^2\right)S^5 - 3S^7\right]\end{aligned}$$

where $c_1 = B(\beta v - \gamma)$, $c_2 = B^2[(1-\lambda)v^2 + \delta v - k^2]$, $c_3 = B^3\sigma v$, and $c_4 = B^4\nu$. From this equation, we can see that c_1 and c_3 must both be zero, and the remaining part can be summarized in the following matrix

$$\begin{pmatrix} 1 & 1 & 1-2A^2 & 1-20A^2+24A^4 \\ 3 & 1 & 1-4A^2 & 1+56A^2-24A^4 \\ 3 & -1 & -1-2A^2 & 21-20A^2 \\ 1 & -1 & -1 & -3 \end{pmatrix}\begin{pmatrix} -a_1 \\ 2a_2 \\ c_2 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

This matrix reduces to

$$\begin{pmatrix} 1 & 0 & 1-A^2 & 0 \\ 0 & 1 & -A^2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which gives the result $c_4 = 0$ and two other relationships that can be used to solve for A and B. The constraints for this equation are those in (2.6) plus

$$A = \pm\sqrt{\frac{a_1}{a_1 - 2a_2}}$$
$$B = \pm\sqrt{\frac{2a_2 - a_1}{k^2 - \delta v - (1-\lambda)v^2}}$$

The first solution to the perturbed DSGE is

$$q(x,t) = 2\arctan\left\{\sqrt{\frac{a_1}{a_1 - 2a_2}}\sinh\left[\pm\sqrt{\frac{2a_2 - a_1}{k^2 - \delta v - (1-\lambda)v^2}}(x - vt)\right]\right\}$$

with the constraints found in (2.6) and $2a_2 > a_1$ and $a_1 < 0$.

Using the second Ansatz, (2.12), we get the equation

$$\begin{aligned} 0 = &-c_2(1+A^2)T(1-T^2)(1+A^2T^2)^2 + \frac{a_1}{2}T(1+A^2T^2)^3 + a_2T(1-A^2T^2)(1+A^2T^2)^2 \\ &+ \frac{c_1}{2}[1-(1-A^2)T^2 - A^2T^4](1+A^2T^2)^2 \\ &+ c_3(1+A^2)(1-T^2)[1-3(1+A^2)T^2 + A^2T^4](1+A^2T^2) \\ &- 4c_4(1+A^2)(1-T^2)\left[(2+3A^2)T - (3+8A^2+3A^4)T^3 + A^2(5+3A^2)T^5 + \frac{1}{2}A^2T^7\right] \end{aligned}$$

where $c_1 = B(\beta v - \gamma)$, $c_2 = B^2[(1-\lambda)v^2 + \delta v - k^2]$, $c_3 = B^3\sigma v$, and $c_4 = B^4\nu$. It is immediately clear that $c_1 = c_3 = c_4 = 0$, which means $\sigma = \nu = 0$. The remaining part can be summarized in the following matrix equation

$$\begin{pmatrix} -1-A^2 & 1 & 1 \\ 1-A^2-2A^4 & 3A^2 & A^2 \\ 2A^2+A^4-A^6 & 3A^4 & -A^4 \\ A^4+A^6 & A^6 & -A^6 \end{pmatrix} \begin{pmatrix} c_2 \\ \frac{1}{2}a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

This matrix reduces to

$$\begin{pmatrix} A^{-2}+2+A^2 & 0 & -1 \\ A^{-2}-A^2 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

These two relationships can be used to solve for A and B

$$A = \sqrt{\frac{a_2+a_1}{a_2-a_1}}$$
$$B = \pm\frac{1}{2}\sqrt{\frac{a_2^2-a_1^2}{a_2\left[k^2-\delta v-(1-\lambda)v^2\right]}}$$

The second solution to the perturbed DSGE is

$$q(x,t) = 2\arctan\left\{\sqrt{\frac{a_2+a_1}{a_2-a_1}}\tanh\left[\pm\frac{1}{2}\sqrt{\frac{a_2^2-a_1^2}{a_2\left[k^2-\delta v-(1-\lambda)v^2\right]}}(x-vt)\right]\right\}$$

with the constraints found in (2.6) and $a_2 > |a_1|$.

Chapter III: TOPOLOGICAL SOLITONS DUE TO THE SINE-GORDON EQUATION WITH FOURTH ORDER DISPERSION

This chapter adds fourth order dispersion to the SG type equations. Equations with a fourth order dispersion term are commonly referred to as Boussinesq type equations by their resemblance to the Boussinesq equation:

$$q_{tt} - q_{xx} + a\left(q^2\right)_{xx} + bq_{xxxx} = 0$$

The Boussinesq equation is an approximate equation for shallow water waves [9], similar to the well-known Korteveg de Vries (KdV) equation [10, 32]. Both equations model solitary waves, or solitons, along the surface of shallow water. For further discussion of the Boussinesq equation, see [13, 15].

3.1 SINE-GORDON EQUATION WITH FOURTH ORDER DISPERSION

The SGE with fourth order dispersion is

$$q_{tt} - k^2 q_{xx} - dq_{xxxx} + a\sin q = 0 \tag{3.1}$$

The Ansatz for (3.1), given by [28], is

$$q(x,t) = 8\arctan\left\{A\exp\left[B(x-vt)\right]\right\} \tag{3.2}$$

The 8 here means that this is in fact a double soliton. This can occur when two identical solitons interact and combine to form a single unit. A deeper discussion of this process can be found in [8]. The governing equation is

$$0 = -B^2(k^2 - v^2)\left(E + E^3 - E^5 - E^7\right) - dB^4\left(E - 23E^3 + 23E^5 - E^7\right) + a\left(E - 7E^3 + 7E^5 - E^7\right)$$

This leads to the system of equations according to the powers of E

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -23 & -7 \\ -1 & 23 & 7 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where $c_1 = -B^2(k^2 - v^2)$ and $c_2 = -dB^4$. Solving the above system leads to the pair of relations

$$B = \pm\sqrt{\frac{\frac{2}{3}a}{k^2 - v^2}}$$
$$v = \pm\sqrt{k^2 - \sqrt{\frac{4}{3}ad}}$$

which put the following two restrictions on the parameters a and d

$$a > 0$$
$$0 < d < \frac{3k^4}{4a}$$

The solution to the unperturbed dispersive SGE is

$$q(x,t) = 8\arctan\left\{\exp\left[\pm\sqrt{\frac{\frac{2}{3}a}{k^2 - v^2}}(x - x_0 - vt)\right]\right\}$$

This result agrees with the results found in [28]. The perturbed dispersive SGE is

$$q_{tt} - k^2 q_{xx} - d q_{xxxx} + a \sin q = R$$

Using the same Ansatz from (3.2), we have

$$\begin{aligned}
0 &= c_1(E + 3E^3 + 3E^5 + E^7) + c_2(E + E^3 - E^5 - E^7) + c_3(E - 5E^3 - 5E^5 + E^7) \\
&\quad + c_4(E - 23E^3 + 23E^5 - E^7) + c_5(E - 7E^3 + 7E^5 - E^7)
\end{aligned}$$

where $c_1 = B(\beta v - \gamma)$, $c_2 = -B^2[k^2 - \delta v - (1-\lambda)v^2]$, $c_3 = B^3\sigma v$, $c_4 = -B^4(d + \nu)$, and $c_5 = a$. Solving the above equation is equivalent to solving the following matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & -5 & -23 & -7 \\ 3 & -1 & -5 & 23 & 7 \\ 1 & -1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the above system leads to the following equalities

$$\begin{aligned}
\sigma &= 0 \\
\gamma &= \beta v \\
B &= \pm\sqrt{\frac{\frac{2}{3}a}{k^2 - \delta v - (1-\lambda)v^2}} \\
v &= \frac{-\delta \pm \sqrt{\delta^2 + 4(1-\lambda)\left(k^2 - \sqrt{\frac{4a}{3}(d+\nu)}\right)}}{2(1-\lambda)}
\end{aligned}$$

which require the following inequalities to hold true

$$\lambda \neq 1$$

$$a > 0$$

$$0 < d + \nu < \frac{3}{4a}\left(k^2 + \frac{\delta^2}{4(1-\lambda)}\right)^2$$

The solution to the perturbed SGE with fourth order dispersion is

$$q(x,t) = 8\arctan\left\{\exp\left[\pm\sqrt{\frac{\frac{2}{3}a}{k^2 - \delta v - (1-\lambda)v^2}}(x - x_0 - vt)\right]\right\}$$

3.2 SINE-COSINE-GORDON EQUATION WITH FOURTH ORDER DISPERSION

The SCGE with fourth order dispersion is

$$q_{tt} - k^2 q_{xx} - d q_{xxxx} + a\sin q + b\cos q = 0 \tag{3.3}$$

The Ansatz for this equation is

$$q(x,t) = 8\arctan\left\{A\exp\left[B(x - vt)\right]\right\} + C \tag{3.4}$$

It is necessary to set $C = -\arctan(b/a)$ in order to eliminate the even exponents caused by the cosine term. Combining (3.3) and (3.4), we have

$$\begin{aligned} 0 = {} & (v^2 - k^2)B^2\left(E + E^3 - E^5 - E^7\right) - dB^4\left(E - 23E^3 + 23E^5 - E^7\right) \\ & + \sqrt{a^2 + b^2}\left(E - 7E^3 + 7E^5 - E^7\right) \end{aligned}$$

Solving the same system of equations from section 3.1 leads to the similar pair of relations

$$B = \pm\sqrt{\frac{\frac{2}{3}\sqrt{a^2+b^2}}{k^2-v^2}}$$
$$v = \pm\sqrt{k^2 - \sqrt{\frac{4}{3}d\sqrt{a^2+b^2}}}$$

which put the following restriction on the parameter d

$$0 < d < \frac{3k^4}{4\sqrt{a^2+b^2}}$$

The solution to the unperturbed dispersive SCGE is

$$q(x,t) = 8\arctan\left\{\exp\left[\pm\sqrt{\frac{\frac{2}{3}\sqrt{a^2+b^2}}{k^2-v^2}}(x-x_0-vt)\right]\right\} - \arctan\left(\frac{b}{a}\right)$$

The perturbed dispersive SCGE is

$$q_{tt} - k^2 q_{xx} - dq_{xxxx} + a\sin q + b\cos q = R$$

Using the previous Ansatz, the governing equation is

$$\begin{aligned} 0 &= c_1(E+3E^3+3E^5+E^7) + c_2(E+E^3-E^5-E^7) + c_3(E-5E^3-5E^5+E^7) \\ &\quad + c_4(E-23E^3+23E^5-E^7) + c_5(E-7E^3+7E^5-E^7) \end{aligned}$$

where $c_1 = B(\beta v - \gamma)$, $c_2 = -B^2[k^2 - \delta v - (1-\lambda)v^2]$, $c_3 = B^3\sigma v$, $c_4 = -B^4(d+\nu)$, and $c_5 = \sqrt{a^2+b^2}$. This leads to the same matrix equation as before, and that implies the following equalities

$$\sigma = 0$$

$$\gamma = \beta v$$

$$B = \pm\sqrt{\frac{\frac{2}{3}\sqrt{a^2+b^2}}{k^2-\delta v-(1-\lambda)v^2}}$$

$$v = \frac{-\delta \pm \sqrt{\delta^2+4(1-\lambda)\left(k^2-\sqrt{\frac{4}{3}(d+\nu)\sqrt{a^2+b^2}}\right)}}{2(1-\lambda)}$$

which require the following inequalities

$$\lambda \neq 1$$

$$0 < d+\nu < \frac{3}{4\sqrt{a^2+b^2}}\left(k^2+\frac{\delta^2}{4(1-\lambda)}\right)^2$$

Hence, the solution to the perturbed dispersive SCGE is

$$q(x,t) = 8\arctan\left\{\exp\left[\pm\sqrt{\frac{\frac{2}{3}\sqrt{a^2+b^2}}{k^2-\delta v-(1-\lambda)v^2}}(x-x_0-vt)\right]\right\} - \arctan\left(\frac{b}{a}\right)$$

3.3 DOUBLE SINE-GORDON EQUATION WITH FOURTH ORDER DISPERSION

The DSGE with fourth order dispersion is

$$q_{tt} - k^2 q_{xx} - d q_{xxxx} + a_1 \sin q + a_2 \sin 2q = 0 \tag{3.5}$$

The Ansatz for this equation, due to [15], is

$$q(x,t) = 4\arctan\left\{A\exp\left[B(x-vt)\right]\right\} \tag{3.6}$$

Inserting (3.6) into (3.5) yields

$$\begin{aligned}0 = &\left[(v^2-k^2)B^2+a_1\right]\left(E+E^3-E^5-E^7\right) - dB^4\left(E-23E^3+23E^5-E^7\right)\\ &+2a_2\left(E-7E^3+7E^5-E^7\right)\end{aligned}$$

This leads to the system of equations according to the powers of E

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -23 & -7 \\ -1 & 23 & 7 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where $c_1 = a_1 - B^2(k^2-v^2)$, $c_2 = -dB^4$, and $c_3 = 2a_2$. Solving the above system leads to the pair of relations

$$\begin{aligned} B &= \pm\sqrt{\frac{a_1+\frac{4}{3}a_2}{k^2-v^2}} \\ v &= \pm\sqrt{k^2-\left(a_1+\frac{4}{3}a_2\right)\sqrt{\frac{3d}{2a_2}}} \end{aligned}$$

which put the following two restrictions on the parameters a_1, a_2, and d

$$\begin{gathered} a_2\cdot d > 0 \\ 0 < a_1+\frac{4}{3}a_2 < k^2\sqrt{\frac{2a_2}{3d}} \end{gathered}$$

The solution to the unperturbed dispersive DSGE is

$$q(x,t) = 4\arctan\left\{\exp\left[\pm\sqrt{\frac{a_1 + \frac{4}{3}a_2}{k^2 - v^2}}(x - x_0 - vt)\right]\right\}$$

The perturbed dispersive DSGE is

$$q_{tt} - k^2 q_{xx} - dq_{xxxx} + a_1 \sin q + a_2 \sin 2q = R$$

Using the Ansatz found in (3.6), the governing equation is

$$\begin{aligned} 0 &= c_1(E + 3E^3 + 3E^5 + E^7) + c_2(E + E^3 - E^5 - E^7) + c_3(E - 5E^3 - 5E^5 + E^7) \\ &\quad + c_4(E - 23E^3 + 23E^5 - E^7) + c_5(E - 7E^3 + 7E^5 - E^7) \end{aligned}$$

where $c_1 = B(\beta v - \gamma)$, $c_2 = a_1 - B^2[k^2 - \delta v - (1-\lambda)v^2]$, $c_3 = B^3\sigma v$, $c_4 = -B^4(d + \nu)$, and $c_5 = 2a_2$. Solving the above equation is equivalent to solving the following matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & -5 & -23 & -7 \\ 3 & -1 & -5 & 23 & 7 \\ 1 & -1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the above system leads to the following equalities

$$\sigma = 0$$

$$\gamma = \beta v$$

$$B = \pm\sqrt{\frac{a_1 + \frac{4}{3}a_2}{k^2 - \delta v - (1-\lambda)v^2}}$$

$$v = \frac{-\delta \pm \sqrt{\delta^2 + 4(1-\lambda)\left[k^2 - \left(a_1 + \frac{4}{3}a_2\right)\sqrt{\frac{3}{2a_2}(d+\nu)}\right]}}{2(1-\lambda)}$$

which require the following inequalities to hold true

$$\lambda \neq 1$$

$$a_1 + \frac{4}{3}a_2 > 0$$

$$0 < \frac{d+\nu}{a_2} < \frac{2}{3}\left(\frac{k^2 + \frac{\delta^2}{4(1-\lambda)}}{a_1 + \frac{4}{3}a_2}\right)^2$$

The solution to the perturbed DSGE with fourth order dispersion is

$$q(x,t) = 8\arctan\left\{\exp\left[\pm\sqrt{\frac{a_1 + \frac{4}{3}a_2}{k^2 - \delta v - (1-\lambda)v^2}}(x - x_0 - vt)\right]\right\}$$

These results agree with those found in [15].

3.4 DOUBLE SINE-COSINE-GORDON EQUATION WITH FOURTH ORDER DISPERSION

The double sine-cosine-Gordon equation (DSCGE) with fourth order dispersion is

$$q_{tt} - k^2 q_{xx} - d q_{xxxx} + a_1 \sin q + b_1 \cos q + a_2 \sin 2q + b_2 \cos 2q = 0 \tag{3.7}$$

The Ansatz for this equation is

$$q(x,t) = 4\arctan\left\{A\exp\left[B(x-vt)\right]\right\} + C \tag{3.8}$$

It is necessary to set $C = -\arctan(b_1/a_1)$ and to ensure

$$\frac{a_2}{b_2} = \frac{a_1^2 - b_1^2}{2a_1b_1}$$

in order to eliminate the even exponents caused by the cosine terms. These imply the pair of simplifications

$$\begin{aligned} a_1\sin q + b_1\cos q &= \sqrt{a_1^2+b_1^2}\sin q \\ a_2\sin 2q + b_2\cos 2q &= \frac{a_2(a_1^2-b_1^2)+2a_1b_1b_2}{a_1^2+b_1^2}\sin 2q \end{aligned}$$

Combining (3.7) and (3.8) gives us the governing equation

$$\begin{aligned} 0 = &\left[\sqrt{a_1^2+b_1^2} - B^2(k^2-v^2)\right]\left(E+E^3-E^5-E^7\right) - dB^4\left(E-23E^3+23E^5-E^7\right) \\ &+ 2\frac{a_2(a_1^2-b_1^2)+2a_1b_1b_2}{a_1^2+b_1^2}\left(E-7E^3+7E^5-E^7\right) \end{aligned}$$

Solving the same system of equations from section 3.3 leads to the similar pair of relations

$$\begin{aligned} B &= \pm\sqrt{\frac{(a_1^2+b_1^2)^{\frac{3}{2}} + \frac{4}{3}\left[a_2(a_1^2-b_1^2)+2a_1b_1b_2\right]}{(k^2-v^2)(a_1^2+b_1^2)}} \\ v &= \pm\sqrt{k^2 - \left(a_1^2+b_1^2+\frac{4}{3}\cdot\frac{a_2(a_1^2-b_1^2)+2a_1b_1b_2}{\sqrt{a_1^2+b_1^2}}\right)\sqrt{\frac{\frac{3}{2}d}{a_2(a_1^2-b_1^2)+2a_1b_1b_2}}} \end{aligned}$$

which put the following restrictions on the parameter a_1, a_2, b_1, b_2, and d

$$d\left[a_2(a_1^2-b_1^2)+2a_1b_1b_2\right]>0$$
$$0<(a_1^2+b_1^2)^{\frac{3}{2}}+\frac{4}{3}\left[a_2(a_1^2-b_1^2)+2a_1b_1b_2\right]$$
$$(a_1^2+b_1^2)^{\frac{3}{2}}+\frac{4}{3}\left[a_2(a_1^2-b_1^2)+2a_1b_1b_2\right]<k^2\sqrt{\frac{(a_1^2+b_1^2)\left[a_2(a_1^2-b_1^2)+2a_1b_1b_2\right]}{\frac{3}{2}d}}$$

The solution to the unperturbed dispersive DSCGE is

$$\begin{aligned} q(x,t) = 4\arctan\left\{\exp\left[\pm\sqrt{\frac{(a_1^2+b_1^2)^{\frac{3}{2}}+\frac{4}{3}\left[a_2(a_1^2-b_1^2)+2a_1b_1b_2\right]}{(k^2-v^2)(a_1^2+b_1^2)}}(x-x_0-vt)\right]\right\} \\ -\arctan\left(\frac{b_1}{a_1}\right) \end{aligned}$$

The perturbed dispersive DSCGE is

$$q_{tt}-k^2q_{xx}-dq_{xxxx}+a_1\sin q+b_1\cos q+a_2\sin 2q+b_2\cos 2q=R$$

Using the Ansatz found in (3.8) and the previously found relationships for C, a_1, b_1, a_2, and b_2, we have

$$\begin{aligned} 0 &= c_1(E+3E^3+3E^5+E^7)+c_2(E+E^3-E^5-E^7)+c_3(E-5E^3-5E^5+E^7) \\ &\quad + c_4(E-23E^3+23E^5-E^7)+c_5(E-7E^3+7E^5-E^7) \end{aligned}$$

where $c_1=B(\beta v-\gamma)$, $c_2=\sqrt{a_1^2+b_1^2}-B^2[k^2-\delta v-(1-\lambda)v^2]$, $c_3=B^3\sigma v$, $c_4=-B^4(d+\nu)$, and $c_5=2[a_2(a_1^2-b_1^2)+2a_1b_1b_2]/(a_1^2+b_1^2)$. The matrix equation is therefore the same as it was in the previous section. This leads to the following equalities

$$\sigma = 0$$

$$\gamma = \beta v$$

$$B = \pm\sqrt{\frac{(a_1^2+b_1^2)^{\frac{3}{2}} + \frac{4}{3}\left[a_2(a_1^2-b_1^2)+2a_1b_1b_2\right]}{\left[k^2-\delta v-(1-\lambda)v^2\right](a_1^2+b_1^2)}}$$

$$v = \frac{1}{2(1-\lambda)}\left(-\delta \pm \left\{\delta^2 + 4(1-\lambda)\left[k^2 - \left(a_1^2+b_1^2+\frac{4}{3}\cdot\frac{a_2(a_1^2-b_1^2)+2a_1b_1b_2}{\sqrt{a_1^2+b_1^2}}\right)\right.\right.\right.$$
$$\left.\left.\left.\times\sqrt{\frac{\frac{3}{2}(d+\nu)}{a_2(a_1^2-b_1^2)+2a_1b_1b_2}}\right]\right\}^{\frac{1}{2}}\right)$$

in which case the following inequalities must also hold true

$$\lambda \neq 1$$
$$\left[a_2(a_1^2-b_1^2)+2a_1b_1b_2\right](d+\nu) > 0$$
$$0 < (a_1^2+b_1^2)^{\frac{3}{2}} + \frac{4}{3}\left[a_2(a_1^2-b_1^2)+2a_1b_1b_2\right]$$
$$a_1^2+b_1^2+\frac{4}{3}\cdot\frac{a_2(a_1^2-b_1^2)+2a_1b_1b_2}{\sqrt{a_1^2+b_1^2}} < \left(k^2+\frac{\delta^2}{4(1-\lambda)}\right)\sqrt{\frac{a_2(a_1^2-b_1^2)+2a_1b_1b_2}{\frac{3}{2}(d+\nu)}}$$

The solution to the perturbed dispersive DSCGE is

$$q(x,t) = 4\arctan\left\{\exp\left[\pm\sqrt{\frac{(a_1^2+b_1^2)^{\frac{3}{2}} + \frac{4}{3}\left[a_2(a_1^2-b_1^2)+2a_1b_1b_2\right]}{\left[k^2-\delta v-(1-\lambda)v^2\right](a_1^2+b_1^2)}}(x-x_0-vt)\right]\right\}$$
$$- \arctan\left(\frac{b_1}{a_1}\right)$$

Chapter IV: TOPOLOGICAL SOLITONS DUE TO THE SINE-GORDON EQUATION WITH SIXTH ORDER DISPERSION

Higher order dispersion terms mainly come about from stronger interactions of the highly discretized SGE, see [11]. Just as the fourth order dispersion term gave way to double solitons, the sixth order dispersion term will yield triple solitons. For further discussion of the discretized SGE and these triple solitons, see [8].

4.1 SINE-GORDON EQUATION WITH SIXTH ORDER DISPERSION

The SGE with sixth order dispersion is

$$q_{tt} - k^2 q_{xx} - d_1 q_{xxxx} - d_2 q_{xxxxxx} + a \sin q = 0 \tag{4.1}$$

The Ansatz for this equation, found in [28], is

$$q(x,t) = 12 \arctan \left\{ A \exp \left[B(x - vt) \right] \right\} \tag{4.2}$$

The 12 here means this is a triple soliton. Plugging (4.2) into (4.1) gives us

$$\begin{aligned} 0 = &-B^2(k^2 - v^2)\left(E + 3E^3 + 2E^5 - 2E^7 - 3E^9 - E^{11}\right) \\ &- d_1 B^4 \left(E - 21E^3 - 22E^5 + 22E^7 + 21E^9 - E^{11}\right) \\ &- d_2 B^6 \left(E - 237E^3 + 1682E^5 - 1682E^7 + 237E^9 - E^{11}\right) \\ &+ \frac{a}{3}\left(3E - 55E^3 + 198E^5 - 198E^7 + 55E^9 - 3E^{11}\right) \end{aligned}$$

This leads to the system of equations according to the powers of E

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 3 & -21 & -237 & -55 \\ 2 & -22 & 1682 & 198 \\ -2 & 22 & -1682 & -198 \\ -3 & 21 & 237 & 55 \\ -1 & -1 & -1 & -3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where $c_1 = -B^2(k^2 - v^2)$, $c_2 = -d_1 B^4$, $c_3 = -d_2 B^6$, and $c_4 = a/3$. Solving the above system leads to the pair of relations

$$B = \pm\sqrt{\frac{\frac{23}{45}a}{k^2 - v^2}}$$
$$v = \pm\sqrt{k^2 - \frac{23}{30}\sqrt{ad_1}}$$

which put the following four restrictions on the parameters a, d_1, and d_2

$$a > 0$$
$$d_2 = \frac{3}{20}\sqrt{\frac{d_1^3}{a}}$$
$$0 < d_1 < \frac{900k^4}{529a}$$

The solution to the unperturbed highly dispersive SGE is

$$q(x,t) = 12\arctan\left\{\exp\left[\pm\sqrt{\frac{\frac{23}{45}a}{k^2 - v^2}}(x - x_0 - vt)\right]\right\}$$

The perturbed SGE with sixth order dispersion is

$$q_{tt} - k^2 q_{xx} - d_1 q_{xxxx} - d_2 q_{xxxxxx} + a \sin q = R$$

Using the same Ansatz as in (4.2) yields

$$\begin{aligned}
0 = {} & B(\beta v - \gamma)\left(E + 5E^3 + 10E^5 + 10E^7 + 5E^9 + E^{11}\right) \\
& - B^2(k^2 - \delta v - (1-\lambda)v^2)\left(E + 3E^3 + 2E^5 - 2E^7 - 3E^9 - E^{11}\right) \\
& + B^3 \sigma v \left(E - 3E^3 - 14E^5 - 14E^7 - 3E^9 + E^{11}\right) \\
& - B^4(d_1 + \nu)\left(E - 21E^3 - 22E^5 + 22E^7 + 21E^9 - E^{11}\right) \\
& - d_2 B^6 \left(E - 237E^3 + 1682E^5 - 1682E^7 + 237E^9 - E^{11}\right) \\
& + \frac{a}{3}\left(3E - 55E^3 + 198E^5 - 198E^7 + 55E^9 - 3E^{11}\right)
\end{aligned}$$

This is equivalent to the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 3 \\ 5 & 3 & -3 & -21 & -237 & -55 \\ 10 & 2 & -14 & -22 & 1682 & 198 \\ 10 & -2 & -14 & 22 & -1682 & -198 \\ 5 & -3 & -3 & 21 & 237 & 55 \\ 1 & -1 & 1 & -1 & -1 & -3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where $c_1 = B(\beta v - \gamma)$, $c_2 = -B^2(k^2 - \delta v - (1-\lambda)v^2)$, $c_3 = B^3\sigma v$, $c_4 = -B^4(d_1+\nu)$, $c_5 = -d_2 B^6$, and $c_6 = a/3$. Solving the above system gives us

$$\sigma = 0$$
$$\gamma = \beta v$$
$$d_2 = \frac{3}{20}\sqrt{\frac{(d_1+\nu)^3}{a}}$$
$$B = \pm\sqrt{\frac{\frac{23}{45}a}{k^2 - \delta v - (1-\lambda)v^2}}$$
$$v = \frac{-\delta \pm \sqrt{\delta^2 + 4(1-\lambda)\left(k^2 - \frac{23}{30}\sqrt{a(d_1+\nu)}\right)}}{2(1-\lambda)}$$

with constraints

$$\lambda \neq 1$$
$$a > 0$$
$$0 < d_1 + \nu < \frac{900}{529a}\left(k^2 + \frac{\delta^2}{4(1-\lambda)}\right)^2$$

The solution to the perturbed highly dispersive SGE is

$$q(x,t) = 12\arctan\left\{\exp\left[\pm\sqrt{\frac{\frac{23}{45}a}{k^2 - \delta v - (1-\lambda)v^2}}(x - x_0 - vt)\right]\right\}$$

4.2 SINE-COSINE-GORDON EQUATION WITH SIXTH ORDER DISPERSION

The SCGE with sixth order dispersion is

$$q_{tt} - k^2 q_{xx} - d_1 q_{xxxx} - d_2 q_{xxxxxx} + a\sin q + b\cos q = 0 \tag{4.3}$$

The Ansatz for this equation is

$$q(x,t) = 12 \arctan \left\{ A \exp \left[B(x - vt) \right] \right\} + C \tag{4.4}$$

First, set $C = -\arctan(b/a)$. Then (4.3) and (4.4) give us

$$\begin{aligned} 0 = &-(k^2 - v^2)B^2 \left(E + 3E^3 + 2E^5 - 2E^7 - 3E^9 - E^{11} \right) \\ &- d_1 B^4 \left(E - 21E^3 - 22E^5 + 22E^7 + 21E^9 - E^{11} \right) \\ &- d_2 B^6 \left(E - 237E^3 + 1682E^5 - 1682E^7 + 237E^9 - E^{11} \right) \\ &+ \frac{1}{3}\sqrt{a^2 + b^2} \left(3E - 55E^3 + 198E^5 - 198E^7 + 55E^9 - 3E^{11} \right) \end{aligned}$$

Solving the same system of equations from section 4.1 leads to the similar pair of relations

$$B = \sqrt{\frac{\frac{23}{45}\sqrt{a^2+b^2}}{k^2 - v^2}}$$

$$v = \sqrt{k^2 - \frac{23}{30}\left(a^2+b^2\right)^{\frac{1}{4}}\sqrt{d_1}}$$

which put the following pair of restrictions on parameters d_1 and d_2

$$d_2 = \frac{3}{20} \cdot \frac{d_1^{\frac{3}{2}}}{\left(a^2+b^2\right)^{\frac{1}{4}}}$$

$$0 < d_1 < \frac{900k^4}{529\sqrt{a^2+b^2}}$$

The solution to the unperturbed highly dispersive SCGE is

$$q(x,t) = 12 \arctan \left\{ \exp \left[\sqrt{\frac{\frac{23}{45}\sqrt{a^2+b^2}}{k^2 - v^2}} (x - x_0 - vt) \right] \right\} - \arctan\left(\frac{b}{a}\right)$$

The perturbed SCGE with sixth order dispersion is

$$q_{tt} - k^2 q_{xx} - d_1 q_{xxxx} - d_2 q_{xxxxxx} + a \sin q + b \cos q = R$$

Using the same Ansatz from (4.4), we have

$$\begin{aligned}
0 = & B(\beta v - \gamma)\left(E + 5E^3 + 10E^5 + 10E^7 + 5E^9 + E^{11}\right) \\
& - B^2(k^2 - \delta v - (1-\lambda)v^2)\left(E + 3E^3 + 2E^5 - 2E^7 - 3E^9 - E^{11}\right) \\
& + B^3 \sigma v \left(E - 3E^3 - 14E^5 - 14E^7 - 3E^9 + E^{11}\right) \\
& - B^4(d_1 + \nu)\left(E - 21E^3 - 22E^5 + 22E^7 + 21E^9 - E^{11}\right) \\
& - d_2 B^6 \left(E - 237E^3 + 1682E^5 - 1682E^7 + 237E^9 - E^{11}\right) \\
& + \frac{1}{3}\sqrt{a^2 + b^2}\left(3E - 55E^3 + 198E^5 - 198E^7 + 55E^9 - 3E^{11}\right)
\end{aligned}$$

This leads to the same matrix equation from section 4.1 where $c_1 = B(\beta v - \gamma)$, $c_2 = -B^2[k^2 - \delta v - (1-\lambda)v^2]$, $c_3 = B^3 \sigma v$, $c_4 = -B^4(d_1 + \nu)$, $c_5 = -d_2 B^6$, and $c_6 = \sqrt{a^2 + b^2}/3$. The relationships now are

$$\begin{aligned}
\sigma &= 0 \\
\gamma &= \beta v \\
d_2 &= \frac{3}{20} \frac{(d_1 + \nu)^{\frac{3}{2}}}{(a^2 + b^2)^{\frac{1}{4}}} \\
B &= \pm \sqrt{\frac{\frac{23}{45}\sqrt{a^2 + b^2}}{k^2 - \delta v - (1-\lambda)v^2}} \\
v &= \frac{-\delta \pm \sqrt{\delta^2 + 4(1-\lambda)\left(k^2 - \frac{23}{30}(a^2 + b^2)^{\frac{1}{4}}\sqrt{d_1 + \nu}\right)}}{2(1-\lambda)}
\end{aligned}$$

with constraints

$$\lambda \neq 1$$
$$0 < d_1 + \nu < \frac{900}{529\sqrt{a^2+b^2}} \left(k^2 + \frac{\delta^2}{4(1-\lambda)}\right)^2$$

The solution to the perturbed highly dispersive SCGE is

$$q(x,t) = 12 \arctan \left\{ \exp \left[\pm \sqrt{\frac{\frac{23}{45}\sqrt{a^2+b^2}}{k^2 - \delta v - (1-\lambda)v^2}} (x - x_0 - vt) \right] \right\} - \arctan\left(\frac{b}{a}\right)$$

4.3 TRIPLE SINE-GORDON EQUATION WITH SIXTH ORDER DISPERSION

The triple sine-Gordon equation (TSGE) with sixth order dispersion is

$$q_{tt} - k^2 q_{xx} - d_1 q_{xxxx} - d_2 q_{xxxxxx} + a_1 \sin q + a_2 \sin 2q + a_3 \sin 3q = 0 \tag{4.5}$$

The Ansatz for (4.5), due to [28], is

$$q(x,t) = 4 \arctan \{A \exp [B(x - vt)]\} \tag{4.6}$$

Plugging (4.6) into (4.5) gives us

$$\begin{aligned} 0 = & \left[a_1 - B^2(k^2 - v^2)\right] \left(E + 3E^3 + 2E^5 - 2E^7 - 3E^9 - E^{11}\right) \\ & - d_1 B^4 \left(E - 21E^3 - 22E^5 + 22E^7 + 21E^9 - E^{11}\right) \\ & - d_2 B^6 \left(E - 237E^3 + 1682E^5 - 1682E^7 + 237E^9 - E^{11}\right) \\ & + 2a_2 \left(E - 5E^3 - 6E^5 + 6E^7 + 5E^9 - E^{11}\right) \\ & + a_3 \left(3E - 55E^3 + 198E^5 - 198E^7 + 55E^9 - 3E^{11}\right) \end{aligned}$$

This leads to the system of equations according to the powers of E

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 3 \\ 3 & -21 & -237 & -5 & -55 \\ 2 & -22 & 1682 & -6 & 198 \\ -2 & 22 & -1682 & 6 & -198 \\ -3 & 21 & 237 & 5 & 55 \\ -1 & -1 & -1 & -1 & -3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where $c_1 = a - B^2(k^2 - v^2)$, $c_2 = -d_1B^4$, $c_3 = -d_2B^6$, $c_4 = 2a_2$, and $c_5 = a_3$. Solving the above system leads to the following relations

$$B = \pm\sqrt{\frac{a_1 + \frac{4}{3}a_2 + \frac{23}{15}a_3}{k^2 - v^2}}$$

$$v = \pm\sqrt{k^2 - \left(a_1 + \frac{4}{3}a_2 + \frac{23}{15}a_3\right)\sqrt{\frac{\frac{3}{2}d_1}{a_2 + 2a_3}}}$$

$$d_2 = \frac{2a_3}{15}\left(\frac{\frac{3}{2}d_1}{a_2 + 2a_3}\right)^{\frac{3}{2}}$$

which put the following four restrictions on the parameters a_1, a_2, a_3, d_1, and d_2

$$d_1(a_2 + 2a_3) > 0$$

$$0 < a_1 + \frac{4}{3}a_2 + \frac{23}{15}a_3 < k^2\sqrt{\frac{a_2 + 2a_3}{\frac{3}{2}d_1}}$$

The solution to the unperturbed highly dispersive TSGE is

$$q(x,t) = 4\arctan\left\{\exp\left[\pm\sqrt{\frac{a_1 + \frac{4}{3}a_2 + \frac{23}{15}a_3}{k^2 - v^2}}(x - x_0 - vt)\right]\right\}$$

The perturbed TSGE with sixth order dispersion is

$$q_{tt} - k^2 q_{xx} - d_1 q_{xxxx} - d_2 q_{xxxxxx} + a_1 \sin q + a_2 \sin 2q + a_3 \sin 3q = R$$

Using the Ansatz found in (4.6) gives

$$\begin{aligned}
0 = &\, B(\beta v - \gamma)\left(E + 5E^3 + 10E^5 + 10E^7 + 5E^9 + E^{11}\right) \\
&+ \left[a_1 - B^2(k^2 - \delta v - (1-\lambda)v^2)\right]\left(E + 3E^3 + 2E^5 - 2E^7 - 3E^9 - E^{11}\right) \\
&+ B^3\sigma v\left(E - 3E^3 - 14E^5 - 14E^7 - 3E^9 + E^{11}\right) \\
&- B^4(d_1 + \nu)\left(E - 21E^3 - 22E^5 + 22E^7 + 21E^9 - E^{11}\right) \\
&- d_2 B^6\left(E - 237E^3 + 1682E^5 - 1682E^7 + 237E^9 - E^{11}\right) \\
&+ 2a_2\left(E - 5E^3 - 6E^5 + 6E^7 + 5E^9 - E^{11}\right) \\
&+ a_3\left(3E - 55E^3 + 198E^5 - 198E^7 + 55E^9 - 3E^{11}\right)
\end{aligned}$$

This is equivalent to the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 3 \\ 5 & 3 & -3 & -21 & -237 & -5 & -55 \\ 10 & 2 & -14 & -22 & 1682 & -6 & 198 \\ 10 & -2 & -14 & 22 & -1682 & 6 & -198 \\ 5 & -3 & -3 & 21 & 237 & 5 & 55 \\ 1 & -1 & 1 & -1 & -1 & -1 & -3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where $c_1 = B(\beta v - \gamma)$, $c_2 = a_1 - B^2[k^2 - \delta v - (1-\lambda)v^2]$, $c_3 = B^3\sigma v$, $c_4 = -B^4(d_1+\nu)$, $c_5 = -d_2 B^6$, $c_6 = 2a_2$, and $c_7 = a_3$. Solving this system results in the following equalities

$$\begin{aligned}
\sigma &= 0 \\
\gamma &= \beta v \\
d_2 &= \frac{2a_3}{15}\left(\frac{3}{2}\cdot\frac{d_1+\nu}{a_2+2a_3}\right)^{\frac{3}{2}} \\
B &= \pm\sqrt{\frac{a_1+\frac{4}{3}a_2+\frac{23}{15}a_3}{k^2-\delta v-(1-\lambda)v^2}} \\
v &= \frac{-\delta \pm \sqrt{\delta^2 + 4(1-\lambda)\left[k^2 - \left(a_1+\frac{4}{3}a_2+\frac{23}{15}a_3\right)\sqrt{\frac{3}{2}\cdot\frac{d_1+\nu}{a_2+2a_3}}\,\right]}}{2(1-\lambda)}
\end{aligned}$$

with constraints

$$\lambda \neq 1$$

$$d_1(a_2+2a_3) > 0$$

$$0 < a_1 + \frac{4}{3}a_2 + \frac{23}{15}a_3 < \sqrt{\frac{2}{3}\cdot\frac{a_2+2a_3}{d_1+\nu}}\left(k^2 + \frac{\delta^2}{4(1-\lambda)}\right)$$

The solution to the perturbed highly dispersive TSGE is

$$q(x,t) = 4\arctan\left\{\exp\left[\pm\sqrt{\frac{a_1+\frac{4}{3}a_2+\frac{23}{15}a_3}{k^2-\delta v-(1-\lambda)v^2}}(x-x_0-vt)\right]\right\}$$

4.4 TRIPLE SINE-COSINE-GORDON EQUATION WITH SIXTH ORDER DISPERSION

The triple sine-cosine-Gordon (TSCGE) equation with sixth order dispersion is

$$\begin{aligned} q_{tt} - k^2 q_{xx} - d_1 q_{xxxx} - d_2 q_{xxxxxx} + a_1 \sin q + b_1 \cos q \\ + a_2 \sin 2q + b_2 \cos 2q + a_3 \sin 3q + b_3 \cos 3q = 0 \end{aligned} \tag{4.7}$$

The Ansatz for (4.7) is

$$q(x,t) = 4 \arctan \{A \exp [B(x - vt)]\} + C \tag{4.8}$$

It is necessary to set $C = -\arctan(b_1/a_1)$ and to ensure both

$$\frac{a_2}{b_2} = \frac{a_1^2 - b_1^2}{2a_1 b_1}$$

and

$$\frac{a_3}{b_3} = -\frac{a_1}{b_1} \cdot \frac{a_1^2 - 3b_1^2}{b_1^2 - 3a_1^2}$$

to eliminate the even exponents of E from the cosine terms. These make the following three simplifications

$$\begin{aligned} a_1 \sin q + b_1 \cos q &= \sqrt{a_1^2 + b_1^2} \sin q \\ a_2 \sin 2q + b_2 \cos 2q &= \frac{a_2(a_1^2 - b_1^2) + 2a_1 b_1 b_2}{a_1^2 + b_1^2} \sin 2q \\ a_3 \sin 3q + b_3 \cos 3q &= \frac{a_1 a_3 (a_1^2 - 3b_1^2) - b_1 b_3 (b_1^2 - 3a_1^2)}{(a_1^2 + b_1^2)^{\frac{3}{2}}} \sin 3q \end{aligned}$$

Combining (4.7) into (4.8) yields

$$
\begin{aligned}
0 = &\left[\sqrt{a_1^2+b_1^2} - B^2(k^2-v^2)\right]\left(E + 3E^3 + 2E^5 - 2E^7 - 3E^9 - E^{11}\right) \\
&- d_1 B^4 \left(E - 21E^3 - 22E^5 + 22E^7 + 21E^9 - E^{11}\right) \\
&- d_2 B^6 \left(E - 237E^3 + 1682E^5 - 1682E^7 + 237E^9 - E^{11}\right) \\
&+ 2\frac{a_2(a_1^2-b_1^2) + 2a_1b_1b_2}{a_1^2+b_1^2}\left(E - 5E^3 - 6E^5 + 6E^7 + 5E^9 - E^{11}\right) \\
&+ \frac{a_1a_3(a_1^2-3b_1^2) - b_1b_3(b_1^2-3a_1^2)}{(a_1^2+b_1^2)^{\frac{3}{2}}}\left(3E - 55E^3 + 198E^5 - 198E^7 + 55E^9 - 3E^{11}\right)
\end{aligned}
$$

This gives rise to the same matrix equation from 4.3, where this time $c_1 = \sqrt{a_1^2+b_1^2} - B^2(k^2-v^2)$, $c_2 = -d_1B^4$, $c_3 = -d_2B^6$, $c_4 = 2[a_2(a_1^2-b_1^2)+2a_1b_1b_2]/(a_1^2+b_1^2)$, and $c_5 = [a_1a_3(a_1^2-3b_1^2) - b_1b_3(b_1^2-3a_1^2)]/(a_1^2+b_1^2)^{\frac{3}{2}}$. The solution to that system gives us

$$
\begin{aligned}
B = \pm &\frac{1}{\sqrt{k^2-v^2}(a_1^2+b_1^2)^{\frac{3}{4}}}\left\{(a_1^2+b_1^2)^2 + \frac{4}{3}\left[a_2(a_1^2-b_1^2)+2a_1b_1b_2\right]\sqrt{a_1^2+b_1^2}\right. \\
&\left. + \frac{23}{15}\left[a_1a_3(a_1^2-3b_1^2) - b_1b_3(b_1^2-3a_1^2)\right]\right\}^{\frac{1}{2}} \\
v = \pm &\left\{k^2 - \left(a_1^2+b_1^2 + \frac{4}{3}\frac{a_2(a_1^2-b_1^2)+2a_1b_1b_2}{\sqrt{a_1^2+b_1^2}} + \frac{23}{15}\frac{a_1a_3(a_1^2-3b_1^2) - b_1b_3(b_1^2-3a_1^2)}{a_1^2+b_1^2}\right)\right. \\
&\left. \times \sqrt{\frac{\frac{3}{2}d_1}{a_2(a_1^2-b_1^2)+2a_1b_1b_2 + 2\frac{a_1a_3(a_1^2-3b_1^2)-b_1b_3(b_1^2-3a_1^2)}{\sqrt{a_1^2+b_1^2}}}}\right\}^{\frac{1}{2}} \\
d_2 = &\frac{2}{15}\left[a_1a_3(a_1^2-3b_1^2) - b_1b_3(b_1^2-3a_1^2)\right]\left(\frac{\frac{3}{2}d_1}{a_2(a_1^2-b_1^2)+2a_1b_1b_2 + 2\frac{a_1a_3(a_1^2-3b_1^2)-b_1b_3(b_1^2-3a_1^2)}{\sqrt{a_1^2+b_1^2}}}\right)^{\frac{3}{2}}
\end{aligned}
$$

and this gives way to the following constraints on the parameters a_1, a_2, a_3, b_1, b_2, b_3,d_1, and d_2

$$d_1\left[a_2(a_1^2-b_1^2)+2a_1b_1b_2+2\frac{a_1a_3(a_1^2-3b_1^2)-b_1b_3(b_1^2-3a_1^2)}{\sqrt{a_1^2+b_1^2}}\right]>0$$

$$0<a_1^2+b_1^2+\frac{4}{3}\frac{a_2(a_1^2-b_1^2)+2a_1b_1b_2}{\sqrt{a_1^2+b_1^2}}+\frac{23}{15}\frac{a_1a_3(a_1^2-3b_1^2)-b_1b_3(b_1^2-3a_1^2)}{a_1^2+b_1^2}$$
$$<k^2\sqrt{\frac{a_2(a_1^2-b_1^2)+2a_1b_1b_2+2\frac{a_1a_3(a_1^2-3b_1^2)-b_1b_3(b_1^2-3a_1^2)}{\sqrt{a_1^2+b_1^2}}}{\frac{3}{2}d_1}}$$

The solution to the unperturbed highly dispersive TSCGE is

$$q(x,t)=4\arctan\left\{\exp\left[\pm\left\{(a_1^2+b_1^2)^2+\frac{4}{3}\left[a_2(a_1^2-b_1^2)+2a_1b_1b_2\right]\sqrt{a_1^2+b_1^2}\right.\right.\right.$$
$$\left.\left.\left.+\frac{23}{15}\left[a_1a_3(a_1^2-3b_1^2)-b_1b_3(b_1^2-3a_1^2)\right]\right\}^{\frac{1}{2}}\frac{(x-x_0-vt)}{(a_1^2+b_1^2)^{\frac{3}{4}}\sqrt{k^2-v^2}}\right]\right\}$$

The perturbed TSCGE with sixth order dispersion is

$$q_{tt}-k^2q_{xx}-d_1q_{xxxx}-d_2q_{xxxxxx}+a_1\sin q+b_1\cos q$$
$$+a_2\sin 2q+b_2\cos 2q+a_3\sin 3q+b_3\cos 3q=R$$

Using the previous Ansatz from (4.8) gives the governing equation

$$
\begin{aligned}
0 &= B(\beta v-\gamma)\left(E+5E^3+10E^5+10E^7+5E^9+E^{11}\right)\\
&+\left[\sqrt{a_1^2+b_1^2}-B^2(k^2-\delta v-(1-\lambda)v^2)\right]\left(E+3E^3+2E^5-2E^7-3E^9-E^{11}\right)\\
&+B^3\sigma v\left(E-3E^3-14E^5-14E^7-3E^9+E^{11}\right)\\
&-B^4(d_1+\nu)\left(E-21E^3-22E^5+22E^7+21E^9-E^{11}\right)\\
&-d_2B^6\left(E-237E^3+1682E^5-1682E^7+237E^9-E^{11}\right)\\
&+2\frac{a_2(a_1^2-b_1^2)+2a_1b_1b_2}{a_1^2+b_1^2}\left(E-5E^3-6E^5+6E^7+5E^9-E^{11}\right)\\
&+\frac{a_1a_3(a_1^2-3b_1^2)-b_1b_3(b_1^2-3a_1^2)}{(a_1^2+b_1^2)^{\frac{3}{2}}}\left(3E-55E^3+198E^5-198E^7+55E^9-3E^{11}\right)
\end{aligned}
$$

This is again equivalent to the matrix equation from 4.3, where this time $c_1 = B(\beta v-\gamma)$, $c_2 = \sqrt{a_1^2+b_1^2}-B^2(k^2-\delta v-(1-\lambda)v^2)$, $c_3 = B^3\sigma v$, $c_4 = -B^4(d_1+\nu)$, $c_5 = -d_2B^6$, $c_6 = 2[a_2(a_1^2-b_1^2)+2a_1b_1b_2]/(a_1^2+b_1^2)$, and $c_7 = [a_1a_3(a_1^2-3b_1^2)-b_1b_3(b_1^2-3a_1^2)]/(a_1^2+b_1^2)^{\frac{3}{2}}$.

Solving that system results in the following equalities

$$
\begin{aligned}
&\sigma = 0 \\
&\gamma = \beta v \\
&d_2 = \frac{2}{15}\left[a_1 a_3(a_1^2 - 3b_1^2) - b_1 b_3(b_1^2 - 3a_1^2)\right] \\
&\qquad \times \left(\frac{\frac{3}{2}(d_1 + \nu)}{a_2(a_1^2 - b_1^2) + 2a_1 b_1 b_2 + 2\frac{a_1 a_3(a_1^2 - 3b_1^2) - b_1 b_3(b_1^2 - 3a_1^2)}{\sqrt{a_1^2 + b_1^2}}} \right)^{\frac{3}{2}} \\
&B = \frac{\pm 1}{\sqrt{k^2 - \delta v - (1-\lambda)v^2}(a_1^2 + b_1^2)^{\frac{3}{4}}} \left\{ (a_1^2 + b_1^2)^2 + \frac{4}{3}\left[a_2(a_1^2 - b_1^2) + 2a_1 b_1 b_2\right] \right. \\
&\qquad \left. \times \sqrt{a_1^2 + b_1^2} + \frac{23}{15}\left[a_1 a_3(a_1^2 - 3b_1^2) - b_1 b_3(b_1^2 - 3a_1^2)\right] \right\}^{\frac{1}{2}} \\
&v = \frac{1}{2(1-\lambda)} \left(-\delta \pm \left\{ \delta^2 + 4(1-\lambda)\left[k^2 - \left(a_1^2 + b_1^2 + \frac{4}{3}\frac{a_2(a_1^2 - b_1^2) + 2a_1 b_1 b_2}{\sqrt{a_1^2 + b_1^2}} \right.\right.\right.\right. \\
&\qquad + \left. \frac{23}{15}\frac{a_1 a_3(a_1^2 - 3b_1^2) - b_1 b_3(b_1^2 - 3a_1^2)}{a_1^2 + b_1^2} \right) \\
&\qquad \left.\left.\left.\left. \times \sqrt{\frac{\frac{3}{2}(d_1 + \nu)}{a_2(a_1^2 - b_1^2) + 2a_1 b_1 b_2 + 2\frac{a_1 a_3(a_1^2 - 3b_1^2) - b_1 b_3(b_1^2 - 3a_1^2)}{\sqrt{a_1^2 + b_1^2}}}} \right] \right\}^{\frac{1}{2}} \right)
\end{aligned}
$$

with constraints

$$
\begin{gathered}
\lambda \neq 1 \\
d_1 \left[a_2(a_1^2 - b_1^2) + 2a_1 b_1 b_2 + 2\frac{a_1 a_3(a_1^2 - 3b_1^2) - b_1 b_3(b_1^2 - 3a_1^2)}{\sqrt{a_1^2 + b_1^2}} \right] > 0 \\
0 < a_1^2 + b_1^2 + \frac{4}{3}\frac{a_2(a_1^2 - b_1^2) + 2a_1 b_1 b_2}{\sqrt{a_1^2 + b_1^2}} + \frac{23}{15}\frac{a_1 a_3(a_1^2 - 3b_1^2) - b_1 b_3(b_1^2 - 3a_1^2)}{a_1^2 + b_1^2} \\
< \left(k^2 + \frac{\delta^2}{4(1-\lambda)} \right) \sqrt{\frac{a_2(a_1^2 - b_1^2) + 2a_1 b_1 b_2 + 2\frac{a_1 a_3(a_1^2 - 3b_1^2) - b_1 b_3(b_1^2 - 3a_1^2)}{\sqrt{a_1^2 + b_1^2}}}{\frac{3}{2}(d_1 + \nu)}}
\end{gathered}
$$

The solution to the perturbed highly dispersive TSCGE is

$$q(x,t) = 4\arctan\left\{\exp\left[\pm\left\{(a_1^2+b_1^2)^2+\frac{4}{3}\left[a_2(a_1^2-b_1^2)+2a_1b_1b_2\right]\sqrt{a_1^2+b_1^2}\right.\right.\right.$$
$$\left.\left.\left.+\frac{23}{15}\left[a_1a_3(a_1^2-3b_1^2)-b_1b_3(b_1^2-3a_1^2)\right]\right\}^{\frac{1}{2}}\frac{(x-x_0-vt)}{(a_1^2+b_1^2)^{\frac{3}{4}}\sqrt{k^2-\delta v-(1-\lambda)v^2}}\right]\right\}$$

Chapter V: 2D SINE GORDON EQUATION

5.1 INTRODUCTION

Many exact solutions have already been found to the 2D SGE [17, 22, 25, 35, 45]. This dissertation presents three new exact solutions. This work was published in 2012 [29]. The 2+1 dimensional SGE studied here is

$$q_{tt} - k_1^2 q_{xx} - k_2^2 q_{yy} + w^2 \sin q = 0 \tag{5.1}$$

In order to match units, k_1 and k_2 are velocities and w is a frequency. The goal here is to find an exact solution to (5.1). The first step is to eliminate the constants k_1, k_2, and w by the change of variables

$$\begin{aligned} t' &= wt \\ x' &= \frac{w}{k_1} x \\ y' &= \frac{w}{k_2} y \end{aligned} \tag{5.2}$$

This makes t', x', and y' all unitless. If $q(x, y, t)$ satisfies (5.1) then that same $q(x', y', t')$ will satisfy the unitless equation

$$q_{t't'} - q_{x'x'} - q_{y'y'} + \sin q = 0 \tag{5.3}$$

5.2 DOMAIN WALL SOLUTION

A standing wave solution to (5.3) is

$$q(x', y', t') = 4 \arctan \{\exp [B_1(x' - x'_0) + B_2(y' - y'_0)]\} \tag{5.4}$$

where B_1 and B_2 are both unitless constants. This solution is called a domain wall. Inserting (5.4) into (5.3) and grouping like terms yields

$$4\frac{E - E^3}{(1 + E^2)^2}\left[-B_1^2 - B_2^2 + 1\right] = 0$$

Where $E(x,t) = \exp\left[B_1(x' - x_0') + B_2(y' - y_0')\right]$. This gives the single relation

$$B_1^2 + B_2^2 = 1 \tag{5.5}$$

Thus either B_1 or B_2 can be a free parameter.

The Lorentz transformation for a unitless "velocity" $\vec{v}'$ with components v_1' in the x' direction and v_2' in the y' direction is

$$\begin{bmatrix} c't'' \\ x'' \\ y'' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta_1\gamma & -\beta_2\gamma \\ -\beta_1\gamma & 1 + (\gamma - 1)\frac{\beta_1^2}{\beta^2} & (\gamma - 1)\frac{\beta_1\beta_2}{\beta^2} \\ -\beta_2\gamma & (\gamma - 1)\frac{\beta_1\beta_2}{\beta^2} & 1 + (\gamma - 1)\frac{\beta_2^2}{\beta^2} \end{bmatrix} \begin{bmatrix} c't' \\ x' \\ y' \end{bmatrix} \tag{5.6}$$

with constants

$$c' = 1$$

$$v' = \sqrt{(v_1')^2 + (v_2')^2}$$

$$\beta = \frac{v'}{c'} = \sqrt{(v_1')^2 + (v_2')^2}$$

$$\beta_1 = \frac{v_1'}{c'} = v_1' \qquad \beta_2 = \frac{v_2'}{c'} = v_2'$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} = \frac{1}{\sqrt{1 - (v_1')^2 - (v_2')^2}}$$

Of course, it is assumed that $(v_1')^2 + (v_2')^2 < 1$. Just as in the case for the 1+1 dimensional SGE, let $x' \to x''$ and $y' \to y''$ under the Lorentz transformation. The standing wave solution then transforms to

$$q(x'', y'', t'') = 4 \arctan \{\exp [B_1(x'' - x_0'') + B_2(y'' - y_0'')]\}$$

The traveling wave solution to (5.3) is

$$\begin{aligned} q(x', y', t') = 4 \arctan \Bigg\{ \exp \Bigg[& \frac{1}{(v')^2} \left(B_1[\gamma(v_1')^2 + (v_2')^2] \right. \\ & \left. + B_2 v_1' v_2' [\gamma - 1] \right) (x' - x_0') + \frac{1}{(v')^2} \left(B_2[(v_1')^2 + \gamma(v_2')^2] \right. \\ & \left. + B_1 v_1' v_2' [\gamma - 1] \right) (y' - y_0') - \gamma (B_1 v_1' + B_2 v_2') t' \Bigg] \Bigg\} \end{aligned} \tag{5.7}$$

Inserting (5.7) into (5.3) leads to the same relation (5.5). While v_1' and v_2' represent the unitless "velocities" in the x' and y' directions respectively, the true velocities in the x and y directions are

$$v_1 = k_1 v_1' \qquad v_2 = k_2 v_2' \tag{5.8}$$

Substituting (5.2) and (5.8) into (5.7) gives the traveling wave solution to (5.1)

$$\begin{aligned} q(x, y, t) = 4 \arctan \Bigg\{ \exp \Bigg[& \frac{k_1^2 k_2^2}{k_1^2 v_2^2 + k_2^2 v_1^2} \Bigg(B_1 \left[\gamma \frac{v_1^2}{k_1^2} + \frac{v_2^2}{k_2^2} \right] \\ & + B_2 \frac{v_1 v_2}{k_1 k_2} [\gamma - 1] \Bigg) \frac{w}{k_1} (x - x_0) + \frac{k_1^2 k_2^2}{k_1^2 v_2^2 + k_2^2 v_1^2} \Bigg(B_2 \left[\frac{v_1^2}{k_1^2} + \gamma \frac{v_2^2}{k_2^2} \right] \\ & + B_1 \frac{v_1 v_2}{k_1 k_2} [\gamma - 1] \Bigg) \frac{w}{k_2} (y - y_0) - \gamma \left(B_1 \frac{v_1}{k_1} + B_2 \frac{v_2}{k_2} \right) wt \Bigg] \Bigg\} \end{aligned} \tag{5.9}$$

where γ is the same value as before

$$\gamma = \frac{1}{\sqrt{1 - \frac{v_1^2}{k_1^2} - \frac{v_2^2}{k_2^2}}}$$

The constraint that $(v_1')^2 + (v_2')^2 < 1$ becomes

$$\frac{v_1^2}{k_1^2} + \frac{v_2^2}{k_2^2} < 1 \tag{5.10}$$

Since x_0 and y_0 are arbitrary constants, t_0 may be added without changing (5.9). Then (5.9) becomes

$$\begin{aligned} q(x,y,t) = 4\arctan\Bigg\{\exp\Bigg[&\frac{wk_1k_2^2}{k_1^2v_2^2 + k_2^2v_1^2}\left(B_1\left[\gamma\frac{v_1^2}{k_1^2} + \frac{v_2^2}{k_2^2}\right]\right. \\ &\left. + B_2\frac{v_1v_2}{k_1k_2}[\gamma - 1]\right)(x - x_0) + \frac{wk_1^2k_2}{k_1^2v_2^2 + k_2^2v_1^2}\left(B_2\left[\frac{v_1^2}{k_1^2} + \gamma\frac{v_2^2}{k_2^2}\right]\right. \\ &\left. + B_1\frac{v_1v_2}{k_1k_2}[\gamma - 1]\right)(y - y_0) - \gamma w\left(B_1\frac{v_1}{k_1} + B_2\frac{v_2}{k_2}\right)(t - t_0)\Bigg]\Bigg\} \end{aligned} \tag{5.11}$$

The fully generalized topological soliton solution to (5.1) is (5.11) with free parameters v_1, v_2, x_0, y_0, t_0, and exactly one of B_1 and B_2 subject to constraints (5.5) and (5.10).

5.3 BREATHER SOLUTION

A second exact solution to (5.1) is called the breather solution. Again, using the change of variables in (5.2), the oscillating but stationary breather solution to (5.3) is

$$q(x', y', t') = 4\arctan\{B_0 \operatorname{sech}[B_1(x' - x_0') + B_2(y' - y_0')]\sin[B_3(t' - t_0')]\} \tag{5.12}$$

assuming $B_0 \neq 0$, $B_3 \neq 0$, and $B_1^2 + B_2^2 \neq 0$. Because B_0 and B_3 both carry a sign, it can be assumed without loss of generality that $B_3 > 0$ to avoid a redundancy. The breather solution (5.12) is topologically different from the domain wall (5.4) in that it flattens out to

0 when $x^2 + y^2 \to \infty$. Inserting (5.12) into (5.3) gives the following two relations

$$B_1^2 + B_2^2 + B_3^2 = 1$$
$$B_0^2 B_3^2 = B_1^2 + B_2^2$$

These two conditions immediately lead to the relations

$$B_3^2 = \frac{1}{1 + B_0^2} \tag{5.13a}$$
$$B_1^2 + B_2^2 = \frac{B_0^2}{1 + B_0^2} \tag{5.13b}$$

which are the 2+1 dimensional analogs of the conditions for the 1+1 dimensional breather. Without loss of generality, we may let B_3 be the positive square root in (5.13a) because the sign of B_3 can be absorbed by B_0 due to the fact that sin is an odd function. The primed coordinates change to the double primed coordinates under the same Lorentz transformation (5.6), leading to

$$q(x'', y'', t'') = 4 \arctan \left\{ B_0 \operatorname{sech} \left[B_1(x'' - x_0'') + B_2(y'' - y_0'') \right] \sin \left[\frac{t'' - t_0''}{\sqrt{1 + B_0^2}} \right] \right\}$$

The moving breather solution to (5.3) is

$$\begin{aligned} q(x', y', t') = 4 \arctan \Bigg\{ B_0 \operatorname{sech} \Bigg[& \frac{x' - x_0'}{(v')^2} \left(B_1[\gamma (v_1')^2 + (v_2')^2] + B_2 v_1' v_2' [\gamma - 1] \right) \\ & + \frac{y' - y_0'}{(v')^2} \left(B_2[(v_1')^2 + \gamma (v_2')^2] + B_1 v_1' v_2' [\gamma - 1] \right) \\ & - \gamma (B_1 v_1' + B_2 v_2')(t' - t_0') \Bigg] \\ & \times \sin \left[\gamma \frac{t' - t_0'}{\sqrt{1 + B_0^2}} - \gamma v_1' \frac{x' - x_0'}{\sqrt{1 + B_0^2}} - \gamma v_2' \frac{y' - y_0'}{\sqrt{1 + B_0^2}} \right] \Bigg\} \end{aligned} \tag{5.14}$$

Substituting (5.2) and (5.8) into (5.14) gives the traveling breather solution to (5.1)

$$\begin{aligned} q(x,y,t) = 4\arctan\Bigg\{ & B_0 \operatorname{sech}\left[\frac{wk_1k_2^2}{k_1^2v_2^2+k_2^2v_1^2}\left(B_1\left[\gamma\frac{v_1^2}{k_1^2}+\frac{v_2^2}{k_2^2}\right]\right.\right. \\ & \left.+B_2\frac{v_1v_2}{k_1k_2}[\gamma-1]\right)(x-x_0)+\frac{wk_1^2k_2}{k_1^2v_2^2+k_2^2v_1^2}\left(B_2\left[\frac{v_1^2}{k_1^2}+\gamma\frac{v_2^2}{k_2^2}\right]\right. \\ & \left.\left.+B_1\frac{v_1v_2}{k_1k_2}[\gamma-1]\right)(y-y_0)-w\gamma\left(B_1\frac{v_1}{k_1}+B_2\frac{v_2}{k_2}\right)(t-t_0)\right] \\ & \times\sin\left[\gamma w\frac{t-t_0}{\sqrt{1+B_0^2}}-\gamma\frac{wv_1}{k_1^2}\frac{x-x_0}{\sqrt{1+B_0^2}}-\gamma\frac{wv_2}{k_2^2}\frac{y-y_0}{\sqrt{1+B_0^2}}\right]\Bigg\} \end{aligned} \tag{5.15}$$

The fully generalized breather solution to (5.1) is (5.15) with free parameters v_1, v_2, x_0, y_0, t_0, B_0, and exactly one of B_1 and B_2 subject to constraints (5.10) and (5.13b).

5.4 DOMAIN WALL COLLISION

A soliton-antisoliton pair solution to (5.3) is

$$q(x',y',t') = 4\arctan\{B_0\operatorname{sech}[B_1(x'-x_0')+B_2(y'-y_0')]\sinh[B_3(t'-t_0')]\} \tag{5.16}$$

again assuming $B_0 \neq 0$, $B_3 \neq 0$, and $B_1^2+B_2^2 \neq 0$. This describes two interacting domain walls that collide with perfect cancellation at $t'=t_0'$. Substituting (5.16) into (5.3) gives the following two relations

$$B_1^2+B_2^2 = B_3^2+1$$

$$B_1^2+B_2^2 = B_0^2B_3^2$$

These two relations can be immediately solved by

$$B_3^2 = \frac{1}{B_0^2 - 1} \tag{5.17a}$$

$$B_1^2 + B_2^2 = \frac{B_0^2}{B_0^2 - 1} \tag{5.17b}$$

It must then be assumed that

$$|B_0| > 1 \tag{5.18}$$

Since B_0 and B_3 both carry a sign, it can again be assumed without loss of generality that $B_3 > 0$. Allowing (5.16) to undergo the same Lorentz transformation in (5.6) yields

$$q(x'', y'', t'') = 4\arctan\left\{B_0 \operatorname{sech}\left[B_1(x'' - x_0'') + B_2(y'' - y_0'')\right] \sinh\left[\frac{t'' - t_0''}{\sqrt{B_0^2 - 1}}\right]\right\}$$

The moving domain wall collision solution to (5.3) is therefore

$$\begin{aligned} q(x', y', t') = 4\arctan\Bigg\{B_0 \operatorname{sech}\Bigg[&\frac{x' - x_0'}{(v')^2}\left(B_1[\gamma(v_1')^2 + (v_2')^2] + B_2 v_1' v_2'[\gamma - 1]\right) \\ &+ \frac{y' - y_0'}{(v')^2}\left(B_2[(v_1')^2 + \gamma(v_2')^2] + B_1 v_1' v_2'[\gamma - 1]\right) \\ &- \gamma(B_1 v_1' + B_2 v_2')(t' - t_0')\Bigg] \\ &\times \sinh\left[\gamma\frac{t' - t_0'}{\sqrt{B_0^2 - 1}} - \gamma v_1' \frac{x' - x_0'}{\sqrt{B_0^2 - 1}} - \gamma v_2' \frac{y' - y_0'}{\sqrt{B_0^2 - 1}}\right]\Bigg\} \end{aligned} \tag{5.19}$$

Substituting (5.2) and (5.8) into (5.19) gives the traveling domain wall collision solution to (5.1)

$$\begin{aligned} q(x,y,t) = 4\arctan\Bigg\{ & B_0 \operatorname{sech}\left[\frac{wk_1k_2^2}{k_1^2v_2^2+k_2^2v_1^2}\left(B_1\left[\gamma\frac{v_1^2}{k_1^2}+\frac{v_2^2}{k_2^2}\right]\right.\right. \\ & \left.+B_2\frac{v_1v_2}{k_1k_2}[\gamma-1]\right)(x-x_0)+\frac{wk_1^2k_2}{k_1^2v_2^2+k_2^2v_1^2}\left(B_2\left[\frac{v_1^2}{k_1^2}+\gamma\frac{v_2^2}{k_2^2}\right]\right. \\ & \left.\left.+B_1\frac{v_1v_2}{k_1k_2}[\gamma-1]\right)(y-y_0)-w\gamma\left(B_1\frac{v_1}{k_1}+B_2\frac{v_2}{k_2}\right)(t-t_0)\right] \\ & \times \sinh\left[\gamma w\frac{t-t_0}{\sqrt{B_0^2-1}}-\gamma\frac{wv_1}{k_1^2}\frac{x-x_0}{\sqrt{B_0^2-1}}-\gamma\frac{wv_2}{k_2^2}\frac{y-y_0}{\sqrt{B_0^2-1}}\right]\Bigg\} \end{aligned} \tag{5.20}$$

The fully generalized domain wall collision to (5.1) is (5.20) with free parameters v_1, v_2, x_0, y_0, t_0, B_0, and exactly one of B_1 and B_2 subject to constraints (5.10), (5.17b), and (5.18).

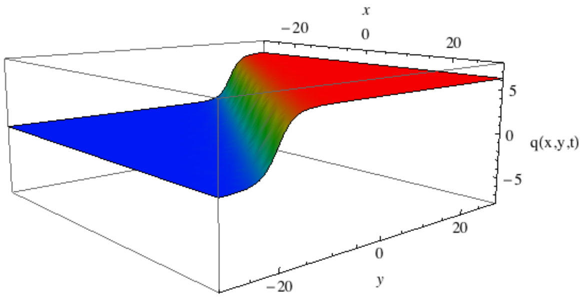
5.5 NUMERICAL SIMULATIONS

In this section we present numerical simulations to the solutions just obtained. These results show similarities between the 2D solutions and 1D solutions.

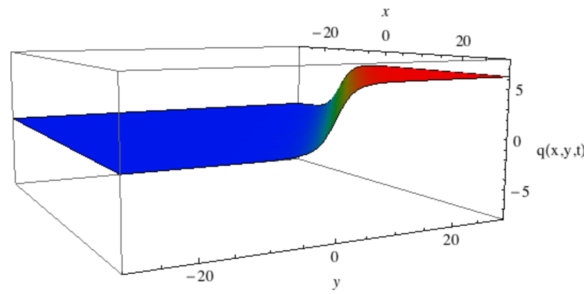
In figure 5.1, the domain wall solution is shown. The parameters chosen are $v_1 = 1$, $v_2 = 1$, $x_0 = 0$, $y_0 = 0$, $t_0 = 0$, $k_1 = 2$, $k_2 = 2$, $B_1 = 1/2$, $B_2 = \sqrt{3}/2$. These numbers were chosen to satisfy the constraints in (5.5) and (5.10).

Figure 5.2 shows the oscillating movement of the Breather in 2D. The 2D breather has a resemblance to the 1D breather. The parameters chosen are $v_1 = 1$, $v_2 = 1$, $x_0 = 0$, $y_0 = 0$, $t_0 = 0$, $k_1 = 2$, $k_2 = 2$, $B_0 = 1$, $B_1 = 1/2$, $B_2 = 1/2$. These parameters were chosen to satisfy the constraints in (5.10), (5.13a), and (5.13b).

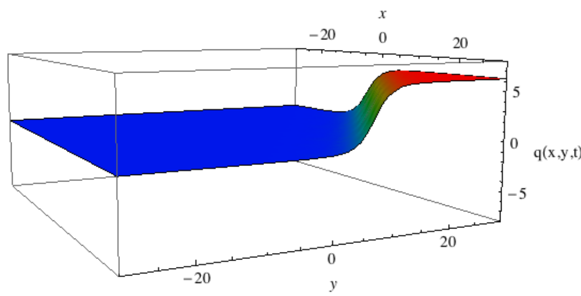
Finally in the domain wall collision we can see the instance just before the wall collision.



(a) Solution at $t = 0$



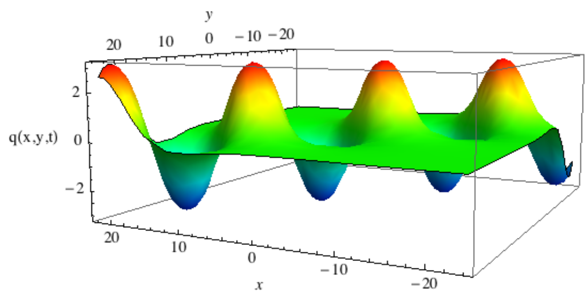
(b) Solution at $t = 6$



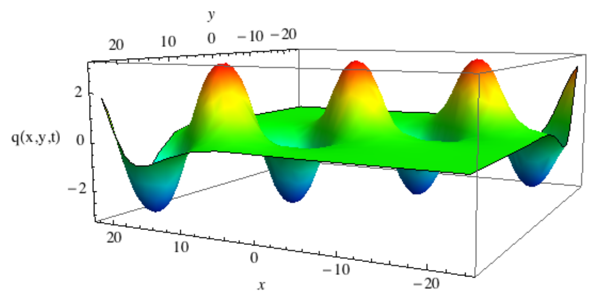
(c) Solution at $t = 10$

Figure 5.1: Domain Wall Solution

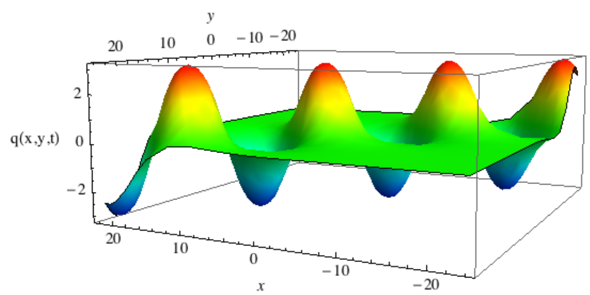
We can see how the waves deform as they collide along the line $y = -x$. The parameters chosen are $v_1 = 1$, $v_2 = 1$, $x_0 = 0$, $y_0 = 0$, $t_0 = 0$, $k_1 = 2$, $k_2 = 2$, $B_0 = 3/2$, $B_1 = 1/2$, $B_2 = \sqrt{31/20}$ and $B_3 = 2/\sqrt{5}$. These parameters were chosen to satisfy the constraints in (5.10), (5.17a), (5.17b), and (5.18).



(a) Solution at $t = 0$

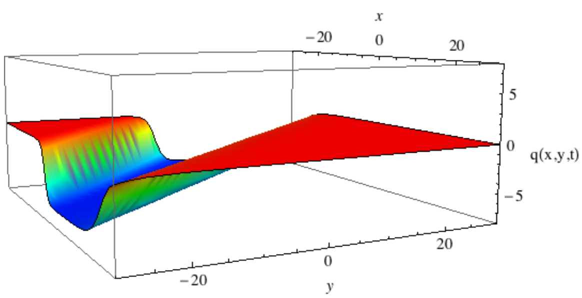


(b) Solution at $t = 1.5$

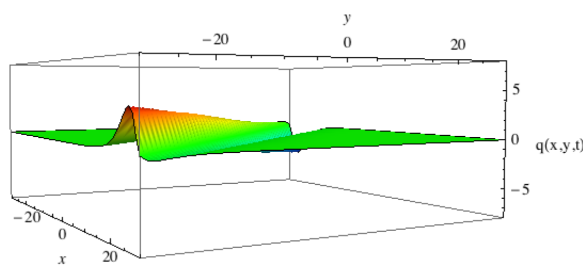


(c) Solution at $t = 2.25$

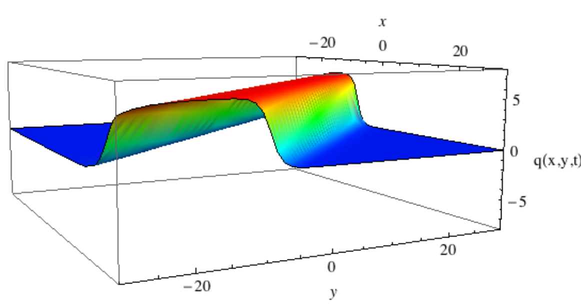
Figure 5.2: Breather Solution



(a) Solution at $t = -6$



(b) Solution at $t = 0$



(c) Solution at $t = 6$

Figure 5.3: Domain Wall Collision

Chapter VI: CONCLUSIONS

This dissertation studied exact solutions to SG type equations, including single, double, and triple sine- and sine-cosine-Gordon equations. These equations included the standard variation, fourth-order dispersion, and sixth-order dispersion. The solutions found are of the topological soliton type, called kinks. After finding the solutions to each unperturbed equation, exact solutions were found for the strongly perturbed variations of each equation. These results will aid in the studies of Josephson junctions, crystal dislocations, ultra-short optical pulses, relativistic field theory, and elementary particles.

Three new generalized traveling wave solutions were found for the 2D SGE. Numerical simulations were presented to corroborate our results. The solutions obtained are natural analogs of solutions to the 1D SGE. It should be emphasized that the methods presented here are straightforward and can be applied to other relativistic solitons and other 2D soliton equations. These results are important for the study of ferromagnetic media and light bullets.

Future work should include the use of numerical methods to further study the solutions found in chapters two through four and possibly find approximate solutions to these equations and other equations of this type. Multiple scale analysis should also be applied to study the effects of weak perturbations on these equations. Future research should also be done to link the particular solutions found in chapter five to observable phenomena.

REFERENCE LIST

[1] A. Barone, F. Esposito, C. J. Magee, and A. C. Scott. Theory and applications of the sine-gordon equation. *La Rivista del Nuovo Cimento*, 1:227–267, 1971.

[2] A. Barone and G. Paterno. *Physics and Applications of the Josephson Effect.* New York: John Wiley & Sons, 1982.

[3] A. H. Bhrawy, M. A. Abdelkawy, S. Kumar, S. Johnson, and A. Biswas. Solitons and other solutions to quantum zakharovkuznetsov equation in quantum magneto-plasmas. *Indian Journal of Physics*, 87:455–463, 2013.

[4] A. Biswas, G. Ebadi, M. Fessak, A. G. Johnpillai, S. Johnson, E. V. Krishnan, and A. Yildirim. Solutions of the perturbed klein-gordon equations. *Iranian Journal of Science & Technology - Transaction A: Science*, 36:431–452, 2012.

[5] A. Biswas, M. Fessak, S. Johnson, B. Siercke, D. Milovic, Z. Jovanoski, R. Kohl, and F. Majid. Optical soliton perturbation in non-kerr law media: traveling wave solution. *Optics & Laser Technology*, 44:263–268, 2012.

[6] A. Biswas, S. Johnson, M. Fessak, B. Siercke, E. Zerrad, and S. Konar. Dispersive optical solitons by the semi-inverse variational principle. *Journal of Modern Optics*, 59:213–217, 2012.

[7] A. Biswas, S. Kumar, E. V. Krishnan, B. Ahmed, A. Strong, S. Johnson, and A. Yildirim. Topological solitons and other solutions to potential korteweg-de vries equation. *Romanian Reports in Physics*, 64:1125–1137, 2013.

[8] M. M. Bogdan, A. M. Kosevich, and G. A. Maugin. Soliton-complex dynamics in strongly dispersive medium. *Wave Motion*, 34:1–26, 2001.

[9] J. Boussinesq. Thorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. *Journal de Mathmatiques Pures et Appliques*, 17:55–108, 1872.

[10] J. Boussinesq. *Essai sur la theorie des eaux courantes*, volume 23 of *Memoires presentes par divers savants a lAcademie des Sciences de lInstitut National de France*. Imprimerie Nationale, Paris, 1877.

[11] M. Chirilus-Bruckner, C. Chong, J. Cuevas-Maraver, and P. G. Kevrekidis. Sine-gordon equation: From discrete to continuum. *Nonlinear Systems and Complexity*, 10:31–57, 2014.

[12] P. G. Drazin and R. S. Johnson. *Solitons: an Introduction*, volume 2 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, 1989.

[13] G. Ebadi, S. Johnson, E. Zerrad, and A. Biswas. Solitons and other nonlinear waves for the perturbed boussinesq equation with power law nonlinearity. *Journal of King Saud University*, 24:237–241, 2012.

[14] G. Ebadi, E. V. Krishnan, S. Johnson, and A. Biswas. Cnoidal wave, snoidal wave, and soliton solutions of the d(m,n) equation. *Arabian Journal of Mathematics*, 2:19–31, 2013.

[15] G. Ebadi, A. Mojaver, S. Johnson, S. Kumar, and A. Biswas. Dynamics of dispersive topological solitons and its perturbations. *Indian Journal of Physics*, 86:1115–1129, 2012.

[16] G. Ebadi, A. Mojaver, D. Milovic, S. Johnson, and A. Biswas. Solitons and other solutions to the quantum zakharov-kuznetsov equation. *Astrophysics and Space Science*, 341:507–513, 2012.

[17] P. G. Estévez and J. Prada. A generalization of the sine-gordon equation to 2+1 dimensions. *Journal of Nonlinear Mathematical Physics*, 11:164–179, 2004.

[18] A. Fabian, R. Kohl, and A. Biswas. Perturbation of topological solitons due to sine-gordon equation and its type. *Communications in Nonlinear Science and Numerical Simulation*, 14:1227–1244, 2009.

[19] S. P. Fitzgerald. Kink pair production and dislocation motion. *Scientific Reports*, 6:39708, 2016.

[20] A. Gharaati, N. Fathi, S. Johnson, and A. Biswas. Analytic solutions of fluxons in a non-homogeneous josephson junction. *Waves in Random and Complex Media*, 22:249–259, 2012.

[21] C. Granata and A. Vettoliere. Nano superconducting quantum interference device: A powerful tool for nanoscale investigations. *Physics Reports*, 614:1–69, 2016.

[22] G. Grella and M. Marinaro. Special solutions of the sine-gordon equation in 2+1 dimensions. *Lettere al Nuovo Cimento*, 23:459–464, 1978.

[23] D. U. Gubser. Navy programs in superconducting technology. *Journal of Superconductivity*, 11:1–4, 1998.

[24] J. V. Guzman, M. F. Mahmood, Q. Zhou, L. Moraru, S. Johnson, A. Biswas, and M. Belic. Singular optical solitons in parabolic and dual-power law media. *Journal of Computational and Theoretical Nanoscience*, 13:4825–4828, 2016.

[25] R. Hirota. Exact three-soliton solution of the two-dimensional sine-gordon equation. *Journal of the Physical Society of Japan*, 35:1566, 1973.

[26] V. G. Ivancevic and T. T. Ivancevic. Sine-gordon solitons, kinks and breathers as physical models of nonlinear excitations in living cellular structures. *Journal of Geometry and Symmetry in Physics*, 31:1–56, 2013.

[27] A. J. M. Jawad, S. Johnson, A. Yildirim, S. Kumar, and A. Biswas. Soliton solutions to coupled nonlinear wave equations in (2+1)-dimensions. *Indian Journal of Physics*, 87:281–287, 2013.

[28] S. Johnson and A. Biswas. Perturbation of dispersive topological solitons. *Physica Scripta*, 84:015002, 2011.

[29] S. Johnson, P. Suarez, and A. Biswas. New exact solutions for the sine-gordon equation in 2+1 dimensions. *Computational Mathematics and Mathematical Physics*, 52:98–104, 2012.

[30] B. D. Josephson. Possible new effects in superconductive tunnelling. *Physics Letters*, 1:251–253, 1962.

[31] Y. Kivshar and B. Malomed. Dynamics of solitons in nearly integrable systems. *Reviews of Modern Physics*, 61:763–915, 1989.

[32] D. J. Korteweg and G. de Vries. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Philosophical Magazine*, 39:422–443, 1895.

[33] N. H. Kuo and C. D. Hu. A study of the solutions of the combined sinecosine-gordon equation. *Applied Mathematics and Computation*, 215:1015–1019, 2009.

[34] H. Leblond and M. Manna. Nonlinear dynamics of two-dimensional electromagnetic solitons in a ferromagnetic slab. *Physical Review B*, 77:224416, 2008.

[35] G. Leibbrandt. New exact solutions of the classical sine-gordon equation in 2+1 and 3+1 dimensions. *Physical Review Letters*, 41:435–438, 1978.

[36] A. Rajantie. Magnetic monopoles in field theory and cosmology. *Philosophical Transactions of the Royal Society A: Mathematical, Physical, and Engineering Sciences*, 370:5705–5717, 2012.

[37] B. Rethfeld, D. S. Ivanov, M. E. Garcia, and S. I. Anisimov. Modeling ultrafast laser ablation. *Journal of Physics D: Applied Physics*, 50:193001, 2017.

[38] M. Savescu, S. Johnson, P. Sanchez, Q. Zhou, M. F. Mahmood, E. Zerrad, A. Biswas, and M. Belic. Nematicons in liquid crystals. *Journal of Computational and Theoretical Nanoscience*, 12:4667–4673, 2015.

[39] P. W. Smith and A. M. Weiner. Ultrashort light pulses. *IEEE Circuits and Devices Magazine*, 4:3–7, 1988.

[40] G. W. Wang, T. Z. Xu, G. Ebadi, S. Johnson, A. Strong, and A. Biswas. Singular solitons, shock waves, and other solutions to potential kdv equation. *Nonlinear Dynamics*, 76:1059–1068, 2014.

[41] G. W. Wang, T. Z. Xu, S. Johnson, and A. Biswas. Solitons and lie group analysis to an extended quantum zakharovkuznetsov equation. *Astrophysics and Space Science*, 349:317–327, 2013.

[42] J. Wattis. Stationary breather modes of generalized nonlinear klein-gordon lattices. *Journal of Physics A: Mathematical and General*, 31:3301–3323, 1998.

[43] A. M. Wazwaz. Travelling wave solutions for combined and double combined sinecosine-gordon equations by the variable separated ode method. *Applied Mathematics and Computation*, 177:755–760, 2006.

[44] J. X. Xin. Modeling light bullets with the two-dimensional sinegordon equation. *Physica D: Nonlinear Phenomena*, 135:345–368, 2000.

[45] J. Zagrodzinsky. Particular solutions of the sine-gordon equation in 2+1 dimensions. *Physics Letters A*, 72:284–286, 1979.